

# Classical and Quantum Gauged Massless Rarita-Schwinger fields

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We show that, in contrast to known results in the massive case, a minimally gauged massless Rarita-Schwinger field yields consistent classical and quantum theories, with a generalized fermionic gauge invariance. To simplify the algebra, we study a two-component left chiral reduction of the massless theory. We formulate the classical theory in both Lagrangian and Hamiltonian form for a general non-Abelian gauging, and analyze the constraints and the Rarita-Schwinger gauge invariance of the action. An explicit wave front calculation for Abelian gauge fields shows that wave-like modes do not propagate with superluminal velocities. The quantized case is studied in gauge covariant radiation gauge and  $\Psi_0 = 0$  gauge for the Rarita-Schwinger field, by both functional integral and Dirac bracket methods. In  $\Psi_0 = 0$  gauge, the constraints have the form needed to apply the Faddeev-Popov method for deriving a functional integral. The Dirac bracket approach in  $\Psi_0 = 0$  gauge yields consistent Hamilton equations of motion, and in covariant radiation gauge leads to anticommutation relations with the correct positivity properties. We discuss relativistic covariance of the anticommutation relations, and of Rarita-Schwinger scattering from an Abelian potential. We note that fermionic gauge transformations are a canonical transformation, but further details of the transformation between different fermionic gauges are left as an open problem.

## I. INTRODUCTION

### A. Motivations and Background

Cancellation of gauge anomalies is a basic requirement for constructing grand unified models, and the usual assumption is that anomalies must cancel among spin  $\frac{1}{2}$  fermion fields. However, a 1985 paper of Marcus [1] showed that in principle an  $SU(8)$  gauge theory can be constructed with spin  $\frac{3}{2}$  Rarita-Schwinger fermions playing a role in anomaly cancelation, and we have recently constructed [2] a family unification model incorporating this observation. Using gauged spin  $\frac{3}{2}$  fields in a grand unification model raises the question of whether such fields admit a consistent quantum, or even classical theory. It is well known, from papers of Johnson and Sudarshan [3] and Velo and Zwanziger [4], and much subsequent literature (see e.g. Deser and Waldron [5]), that

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theories of massive gauged Rarita-Schwinger fields have serious problems. Does setting the fermion mass to zero eliminate these difficulties?

The lesson we have learned from the success of the Standard Model is that fundamental fermion masses lead to problems and are to be avoided; all mass is generated by spontaneous symmetry breaking, either through coupling to the Higgs boson or through the formation of chiral symmetry breaking fermion condensates. So from a modern point of view, the Rarita-Schwinger theory with an explicit mass term is suspect. Several hints that the behavior of the massless theory may be satisfactory are already apparent from a study of the zero mass limit of formulas in the Velo-Zwanziger paper. First, in their demonstration of superluminal signaling, the problematic sign change that they find for large  $B$  fields (Eq. (2.15) of [4]) is not present when the mass is set to zero. Second, when the mass is zero, the secondary constraint that they derive (Eq. (2.10) of [4]) appears as a factor in the change in the action under a Rarita-Schwinger gauging  $\delta\psi_\mu = D_\mu\epsilon$ , with  $D_\mu$  the usual gauge covariant derivative. (This statement is not in [4], but is an easy calculation from their Eqs. (2.1)–(2.3), with the  $D_\mu$  of this paper their  $-i\pi_\mu$ .) Hence, the constrained action in the massless gauged Rarita-Schwinger theory has a fermionic gauge invariance that is the natural generalization of the fermionic gauge invariance of the massless free Rarita-Schwinger theory. Third, their formula for the anticommutator (Eq. (4.12) of [4]) in the zero mass case develops an apparent singularity in the limit of vanishing gauge field  $B$ , and so their quantization does not limit to the standard free theory quantization. This should not be surprising: since the massive theory does not have a fermionic gauge invariance, Ref. [4] does not include a gauge-fixing term analogous to that used in the massless case, but gauge fixing is needed to get a consistent quantum theory for a free massless Rarita-Schwinger field. So these observations, following from the equations in [4], suggest that a study of the massless Rarita-Schwinger field coupled to spin-1 gauge fields is in order.

In a different setting, massless Rarita-Schwinger fields appear consistently coupled to gravity as the gravitinos of supergravity, as discussed by Das and Freedman [6]. Grisaru, Pendleton and van Nieuwenhuizen [7] have shown that soft spin  $\frac{3}{2}$  fermions *must* be coupled to gravity as in supergravity, in an analysis based on the free particle external line pole structure of spin  $\frac{3}{2}$  fields that do not have spin 1 gauge couplings. Their result does not rule out the possibility of gauged spin  $\frac{3}{2}$  fields with non-Abelian gauge couplings, since such fermion fields will in general not have free particle external lines. For example, a gravitino coupled in the fundamental representation of an unbroken non-Abelian gauge group will be confined, and will not have the kind of external line pole structure used in the argument of [7]. Thus, the known connection of ungauged Rarita-Schwinger

fields to supergravity does not argue against the possibility that there could be a consistent theory of massless, gauged Rarita-Schwinger fields, so again a detailed study of this possibility is warranted.

## B. Outline of the paper, and summary

With these motivations and background in mind, we embark in this paper on a detailed study of the classical and quantum theory of a minimally gauged massless Rarita-Schwinger field. In Sec. 2, we give the Lorentz covariant Lagrangian for a gauged four-component Rarita-Schwinger spinor field, derive the source current for the gauge field, and check that it is gauge-covariantly conserved. We also give the Lorentz covariant form of the constraints and of the fermionic gauge invariance, and of the symmetric stress-energy tensor. In Sec. 3 we argue that the massless, gauged Rarita-Schwinger theory exhibits a generalized fermionic gauge invariance, in the sense that the action principle does not uniquely determine the system time evolution. Since in the massless case left chiral and right chiral components of the field decouple, in Sec. 4 we rewrite the Lagrangian for left chiral components in terms of two-component spinors and Pauli matrices, which simplifies the subsequent analysis. We then give the Euler-Lagrange equations in two-component form, and use them to analyze the structure of constraints and the fermionic gauge invariance of the action. In Sec. 5 we specialize to the case of an Abelian gauge field (as in [4]) and analyze the wavefront structure, showing that wave modes propagate with subluminal velocities; an extension of this discussion is given in Appendix B. In Sec. 6 we return to the general case of non-Abelian gauge fields. We introduce canonical momenta for the Rarita-Schwinger field components, which are used to define classical Poisson brackets, and discuss the role of the constraints as generators of gauge transformations under the bracket operation. We show that the constraints group into two sets of four, within each of which there are vanishing Poisson brackets. This permits application of the Faddeev–Popov method for path integral quantization, which we carry through in detail in Sec. 7 in  $\Psi_0 = 0$  gauge. In Sec. 8 we give the Hamiltonian form of the equations of motion and constraints, and introduce the Dirac bracket. When a gauge fixing constraint is omitted, the Dirac bracket that we compute agrees with the anticommutator calculated in [3] and [4]. In Sec. 9 we study the Dirac bracket in its classical and quantum forms, and show in covariant radiation gauge that the quantum Dirac bracket has the requisite positivity properties to be an anticommutator; related details are given in Appendix C. In Sec. 10 we study the behavior under Lorentz transformations of covariant radiation gauge, of  $\Psi_0 = 0$  gauge, and of the Dirac bracket and anticommutator. In Sec. 11 we analyze Rarita-Schwinger fermion scattering from a short range Abelian gauge potential, using an

analog of the distorted wave Born approximation. We show consistency of relativistic covariance with helicity state counting, but there is no small-coupling expansion giving an unmodified Born approximation for this process. In Sec. 12 we discuss areas for extensions of our results, and in Appendix A we summarize our notational conventions and some useful identities. We suggest that the reader skim through Appendix A before going on to Sec. 2, since things stated in Appendix A are not repeated in the body of the paper.

Our conclusion from this analysis is that one can consistently gauge a massless Rarita-Schwinger field, at both the classical and quantum levels. This opens the possibility of using gauged Rarita-Schwinger fields as part of the anomaly cancelation mechanism in grand unified models, with anomalies of the spin  $\frac{1}{2}$  fields canceling against the spin  $\frac{3}{2}$  anomaly.

## II. LAGRANGIAN AND COVARIANT CURRENT CONSERVATION IN FOUR-COMPONENT FORM

The action for the massless Rarita-Schwinger theory is

$$\begin{aligned}
 S(\psi_\mu) &= \frac{1}{2} \int d^4x \bar{\psi}_{\mu\alpha u} R^{\mu\alpha u} \quad , \\
 R^{\mu\alpha u} &= i\epsilon^{\mu\eta\nu\rho} (\gamma_5 \gamma_\eta)^\alpha_\beta (D_\nu \psi_\rho^\beta)^u \quad , \\
 (D_\nu \psi_\rho^\beta)^u &\equiv \partial_\nu \psi_\rho^{\beta u} + g A_{\nu v}^u \psi_\rho^{\beta v} \quad , \\
 A_{\nu v}^u &= A_\nu^A t_{Av}^u \quad ,
 \end{aligned} \tag{1}$$

with  $\psi^{\mu\alpha u}$  a four-vector four-component spinor, with four-vector index  $\mu = 0, \dots, 3$ , spinor index  $\alpha = 1, \dots, 4$ , and  $SU(n)$  internal symmetry index  $u = 1, \dots, n$ , with  $SU(n)$  gauge generators  $t_A$ ,  $A = 1, \dots, n^2 - 1$ . Taking  $u$  to range from 1 to  $n$  means that, for definiteness, we are assuming that the spinors transform according to the fundamental representation of the  $SU(n)$  internal symmetry group, but other representations and other compact Lie groups can be accommodated by assigning the internal indices  $u$  and  $A$  the appropriate range. Note that  $t_A$ ,  $A_{\nu v}^u$ , and  $D_\nu$  all commute with the gamma matrices and the Pauli spin matrices from which the gamma matrices are constructed, and for an Abelian internal symmetry group, the indices  $u$  and  $A$  are not needed. Using

$$\bar{\psi}_{\mu\alpha u} = \psi_{\mu\beta u}^\dagger i(\gamma^0)^\beta_\alpha \quad , \tag{2}$$

together with the adjoint convention  $(\chi_1^\dagger \chi_2)^\dagger = \chi_2^\dagger \chi_1$  for Grassmann variables  $\chi_1, \chi_2$ , it is easy to verify that  $S$  is self-adjoint.

From here on we will usually not indicate the spinor indices  $\alpha, \beta$  and internal symmetry indices  $u, v$  explicitly, but they are implicit in all formulas. Varying  $S$  with respect to the Rarita-Schwinger fields, we get the equations of motion

$$\begin{aligned}\epsilon^{\mu\eta\nu\rho}\partial_\nu\bar{\psi}_\rho\gamma_\eta &= g\epsilon^{\mu\eta\nu\rho}\bar{\psi}_\rho A_\nu^A t_A \gamma_\eta \quad , \\ \epsilon^{\mu\eta\nu\rho}\gamma_\eta\partial_\nu\psi_\rho &= -g\epsilon^{\mu\eta\nu\rho}\gamma_\eta A_\nu^A t_A \psi_\rho \quad .\end{aligned}\tag{3}$$

Re-expressed in terms of the covariant derivative, these are

$$\begin{aligned}\epsilon^{\mu\eta\nu\rho}\bar{\psi}_\rho\overleftarrow{D}_\nu\gamma_\eta &= 0 \quad , \\ \epsilon^{\mu\eta\nu\rho}\gamma_\eta D_\nu\psi_\rho &= 0 \quad .\end{aligned}\tag{4}$$

The  $\mu = 0$  component of these equations gives the primary constraints

$$\begin{aligned}\epsilon^{enr}\bar{\psi}_r\overleftarrow{D}_n\gamma_e &= 0 \quad , \\ \epsilon^{enr}\gamma_e D_n\psi_r &= 0 \quad ,\end{aligned}\tag{5}$$

with  $e, n, r$  summed from 1 to 3. Contracting the equation of motion for  $\bar{\psi}_\rho$  with  $g^{-1}\overleftarrow{D}_\mu$  and the equation of motion for  $\psi_\rho$  with  $g^{-1}D_\mu$ , we get the secondary constraints

$$\begin{aligned}\epsilon^{\mu\eta\nu\rho}\bar{\psi}_\rho F_{\mu\nu}\gamma_\eta &= 0 \quad , \\ \epsilon^{\mu\eta\nu\rho}\gamma_\eta F_{\mu\nu}\psi_\rho &= 0 \quad ,\end{aligned}\tag{6}$$

where we have introduced the gauge field strength

$$\begin{aligned}F_{\mu\nu} &= g^{-1}[D_\mu, D_\nu] = g^{-1}[\overleftarrow{D}_\mu, \overleftarrow{D}_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] \quad ,\end{aligned}\tag{7}$$

which with the adjoint representation index  $A$  indicated explicitly reads

$$F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + gf_{ABC}A_\mu^B A_\nu^C \quad .\tag{8}$$

Under a Rarita-Schwinger gauge transformation (with  $\epsilon$  a four-component spinor), which is a natural gauge field generalization of the fermionic gauge invariance for a free, massless Rarita-Schwinger field discussed in [8],

$$\begin{aligned}\psi_\mu &\rightarrow \psi_\mu + \delta_G \psi_\mu \quad , \quad \delta_G \psi_\mu \equiv D_\mu \epsilon \quad , \\ \bar{\psi}_\mu &\rightarrow \bar{\psi}_\mu + \delta_G \bar{\psi}_\mu \quad , \quad \delta_G \bar{\psi}_\mu \equiv \bar{\epsilon} \overleftarrow{D}_\mu \quad ,\end{aligned}\tag{9}$$

the action of Eq. (1) changes according to

$$\delta_G S(\psi_\mu) = -\frac{1}{4}ig \int d^4x \left[ \bar{\epsilon} \gamma_5 \left( \epsilon^{\mu\nu\rho} \gamma_\eta F_{\mu\nu} \psi_\rho \right) + \left( \epsilon^{\mu\nu\rho} \bar{\psi}_\rho F_{\mu\nu} \gamma_\eta \right) \gamma_5 \epsilon \right] \quad .\tag{10}$$

The factors bracketed in large parentheses are identical to the secondary constraints of Eq. (6). We will argue in the next section that this implies that, even when coupled to gauge fields, the Rarita-Schwinger theory has a generalized form of fermionic gauge invariance.

Adding the gauge field action

$$S(A_\mu^A) = -\frac{1}{4} \int d^4x F_{\mu\nu}^A F^{A\mu\nu} \quad ,\tag{11}$$

and varying the sum  $S(\psi_\mu) + S(A_\mu^A)$  with respect to the gauge potential, we get the gauge field equation of motion

$$\begin{aligned}D_\nu F^{A\mu\nu} &\equiv \partial_\nu F^{A\mu\nu} + g f_{ABC} A_\nu^B F^{C\mu\nu} = g J^{A\mu} \quad , \\ J^{A\mu} &= \frac{1}{2} \bar{\psi}_\nu i \epsilon^{\nu\eta\mu\rho} \gamma_5 \gamma_\eta t_A \psi_\rho \quad .\end{aligned}\tag{12}$$

A straightforward calculation using Eqs. (3) shows that the gauge field source current  $J^{A\mu}$  obeys the covariant conservation equation

$$D_\mu J^{A\mu} = \partial_\mu J^{A\mu} + g f_{ABC} A_\mu^B J^{C\mu} = 0 \quad ,\tag{13}$$

as required for consistency of Eq. (12). So from the Rarita-Schwinger and gauge field actions, we have obtained a formally consistent set of equations of motion.

In addition to the gauge field source current, there is an additional current  $J^\mu$  that obeys an ordinary conservation equation,

$$\begin{aligned}J^\mu &= \frac{1}{2} \bar{\psi}_\nu \epsilon^{\nu\eta\mu\rho} \gamma_5 \gamma_\eta \psi_\rho \quad , \\ \partial_\mu J^\mu &= 0 \quad .\end{aligned}\tag{14}$$

In the massive spinor case, Velo and Zwanziger [4] argue that the analogous current, within the constraint subspace of Eq. (5), should have a positive time component. In the massless case we see no reason for this requirement, since Eq. (14) is the fermion number current, and its time component, giving the fermion number density, can have either sign. However, we shall use parts of the positivity argument of [4] later on in discussing positivity of the Dirac bracket anticommutator.

The symmetric stress-energy tensor for the free massless Rarita-Schwinger has been computed by Das [9] (see also Allcock and Hall [10]). Changing ordinary derivatives to gauge covariant derivatives, Das's formula becomes

$$T_{\text{RS}}^{\sigma\tau} = -\frac{i}{4}\epsilon^{\lambda\mu\nu\rho}\left[\bar{\psi}_\lambda\gamma_5(\gamma^\tau\delta_\mu^\sigma + \gamma^\sigma\delta_\mu^\tau)D_\nu\psi_\rho + \frac{1}{4}\partial_\alpha\left(\bar{\psi}_\lambda\gamma_5\gamma_\mu([\gamma^\alpha, \gamma^\sigma]\delta_\nu^\tau + [\gamma^\alpha, \gamma^\tau]\delta_\nu^\sigma)\psi_\rho\right)\right] . \quad (15)$$

(This formula can be made manifestly self-adjoint by replacing  $D_\nu$  by  $\frac{1}{2}(D_\nu - \overleftarrow{D}_\nu)$ , but this is not needed to verify stress-energy tensor conservation.) Adding the gauge field stress-energy tensor,

$$T_{\text{gauge}}^{\sigma\tau} = -\frac{1}{4}\eta^{\sigma\tau}F_{\lambda\mu}^A F^{A\lambda\mu} + F_\lambda^{A\sigma} F^{A\lambda\tau} , \quad (16)$$

a lengthy calculation, using Eq. (13) together with identities and alternative forms of the equations of motion given in Appendix A, shows that the total tensor is conserved,

$$\partial_\sigma(T_{\text{RS}}^{\sigma\tau} + T_{\text{gauge}}^{\sigma\tau}) = 0 . \quad (17)$$

### III. GENERALIZED GAUGE INVARIANCE OF THE RARITA-SCHWINGER ACTION

In the most familiar gauge invariant theories, such as Abelian or non-Abelian gauge fields, the Lagrangian density is invariant under a gauge transformation on the fields. These theories exhibit what one could term “strong” gauge invariance. In a weaker form of gauge invariance, which occurs for the free Rarita-Schwinger equation, the Lagrangian density changes by a total derivative under a gauge transformation of the fields, and so only the action is gauge invariant. Characteristic features of this case have been studied by Das [9]. We argue in this section that there is a still weaker form of gauge invariance, obeyed by the massless Rarita-Schwinger equation with Abelian or non-Abelian gauging, in which under a gauge transformation the Lagrangian changes by a total derivative plus terms which vanish when initial value constraints are obeyed.

In his seminal analysis of constrained systems, Dirac [11] classifies as “first class” constraints the maximal set of constraints that have vanishing mutual Poisson brackets, and notes that “Each

of them thus leads to an arbitrary function of the time in the general solution of the equations of motion with given initial conditions". Elaborating on this, he notes that "Different solutions of the equations of motion, obtained by different choices of the arbitrary functions of the time with given initial conditions, should be looked upon as all corresponding to the same physical state of motion, described in various way (sic) by different choices of some mathematical variables that are not of physical significance (e.g. by different choices of the gauge in electrodynamics or of the co-ordinate system in a relativistic theory.)"

These remarks suggest that gauge invariance, in its most general form, corresponds to an arbitrariness in the time evolution of a system, in the sense that the future evolution of the system is not uniquely determined by the initial conditions and the Euler-Lagrange equations following from the action principle. Under this generalized definition, the Rarita-Schwinger equation with coupling to gauge fields has a fermionic gauge invariance. To see this, we note the Euler-Lagrange equations yield equations of two types. The first are the time evolution equations contained in Eq. (3), that determine the field variables at time  $t + \Delta t$  from those initially given at time  $t$ . Since we will see that  $\psi_0$  is related by a constraint to  $\vec{\psi}$ , the evolution equations take the form

$$\vec{\psi}(t + \Delta t) = \vec{\psi}(t) + \Delta t \frac{\partial \vec{\psi}}{\partial t} [\vec{\psi}(t), \psi_0(t)] + O((\Delta t)^2) \quad . \quad (18)$$

The second are the primary and secondary constraints of Eqs. (5) and (6), which constrain the initial field values at time  $t$ . If we make the gauge transformation of Eq. (9) at time  $t$ , with gauge parameter  $\epsilon$  of order  $\Delta t$ , we have seen that the action at time  $t$  changes, to first order in  $\epsilon$ , according to Eq. (10). So when the constraints at time  $t$  are applied, the change in the action is  $O((\Delta t)^2)$ . The effect of the gauge shift  $\delta_G \vec{\psi}(t)$  of order  $\Delta t$  is to change the time evolution equation Eq. (18) to

$$\begin{aligned} \vec{\psi}(t + \Delta t) &= \vec{\psi}(t) + \delta_G \vec{\psi}(t) + \Delta t \frac{\partial \vec{\psi}}{\partial t} [\vec{\psi}(t) + \delta_G \vec{\psi}(t), \psi_0(t) + \delta_G \psi_0(t)] + O((\Delta t)^2) \\ &= \vec{\psi}(t) + \delta_G \vec{\psi}(t) + \Delta t \frac{\partial \vec{\psi}}{\partial t} [\vec{\psi}(t), \psi_0(t)] + O((\Delta t)^2) \quad . \end{aligned} \quad (19)$$

In other words, the error induced in the  $\Delta t \partial \vec{\psi} / \partial t$  term by the gauge transformation is  $O((\Delta t)^2)$ . Hence the gauge transformed variables are equally good candidates for time evolution to  $t + \Delta t$ , leading to a state of the system at  $t + \Delta t$  that is no longer unique; for each independent gauge change at time  $t$  we get a different set of evolved variables at time  $t + \Delta t$ . One can now evolve the variables at  $t + \Delta t$  to time  $t + 2\Delta t$ , subject to the constraints at time  $t + \Delta t$ , allowing one to



make a further independent gauge transformation at time  $t + \Delta t$ , and thus introducing a further non-uniqueness into the time-evolved solution. Continuing in this fashion, forming a Riemann sum in the limit  $\Delta t \rightarrow 0$ , the  $O((\Delta t)^2)$  corrections drop out, and so there is no change in the action associated with the gauge-modified system evolution. Thus one can introduce arbitrary functions of time into the evolved solution, corresponding to each independent fermionic gauge transformation of Eq. (9). In order to get a unique time evolution path from the initial data at time  $t$ , one must impose a gauge fixing condition, that selects at each time one member out of the equivalence class of equal action configurations.

Therefore, even though only the constrained action in the gauged Rarita-Schwinger theory is invariant under fermionic gauge transformations, we shall refer to this as gauge-invariance in the sense just described, and will impose a gauge fixing condition to break this invariance. The gauge fixing condition will also serve to give the correct helicity counting for the Rarita-Schwinger fields, as we shall see in detail below. We shall also discuss below how the first class constraints, which follow from repeatedly differentiating the original primary constraints with respect to time, play a role as gauge transformation generators, in agreement with what is found in strongly gauge invariant systems. However, we shall see that a novel feature emerges in the Rarita-Schwinger theory: When one insists that for each first class fermionic constraint there should be a corresponding adjoint constraint, doubling the constraint count of the theory, then the first class constraints become second class constraints in the Dirac classification.

#### IV. LAGRANGIAN ANALYSIS FOR LEFT CHIRAL SPINORS IN TWO-COMPONENT FORM

Although we could continue with the four-component formalism to study constraints, the Hamiltonian formalism, and quantization, it will be more convenient to first reduce the four component equation to decoupled equations for left and right chiral components of  $\psi_\mu^\alpha$  (with  $\alpha$  the spinor index and with the internal symmetry index implicit). Since these are related by symmetry, we can then focus our analysis on the two-component equations for the left chiral component, which is the component conventionally used in formulating grand unified models (see, e.g. [2]).

We convert the action of Eq. (1) to two-component form for the left chiral components of  $\psi_\mu^\alpha$ , using the Dirac matrices given in Eqs. (A2) and (A4). Defining the two-component four vector

spinor  $\Psi_\mu^\alpha$  and its adjoint  $\Psi_{\mu\alpha}^\dagger$  by

$$\begin{aligned} P_L \psi_\mu^\alpha &= \begin{pmatrix} \Psi_\mu^\alpha \\ 0 \end{pmatrix}, \quad \mu = 0, 1, 2, 3, \quad \alpha = 1, 2, \\ \psi_{\mu\alpha}^\dagger P_L &= \begin{pmatrix} \Psi_{\mu\alpha}^\dagger & 0 \end{pmatrix}, \end{aligned} \quad (20)$$

the action decomposes into uncoupled left and right chiral parts. The left chiral part, with spinor indices  $\alpha$  suppressed, is given by

$$S(\Psi_\mu) = \frac{1}{2} \int d^4x [-\Psi_0^\dagger \vec{\sigma} \cdot \vec{D} \times \vec{\Psi} + \vec{\Psi}^\dagger \cdot \vec{\sigma} \times \vec{D} \Psi_0 + \vec{\Psi}^\dagger \cdot \vec{D} \times \vec{\Psi} - \vec{\Psi}^\dagger \cdot \vec{\sigma} \times D_0 \vec{\Psi}] \quad (21)$$

Varying with respect to  $\vec{\Psi}^\dagger$  we get the Euler-Lagrange equation

$$0 = \vec{V} \equiv \vec{\sigma} \times \vec{D} \Psi_0 + \vec{D} \times \vec{\Psi} - \vec{\sigma} \times D_0 \vec{\Psi}, \quad (22)$$

while varying with respect to  $\Psi_0^\dagger$  we get the primary constraint (given in four-component form in Eq. (5))

$$0 = \chi \equiv \vec{\sigma} \cdot \vec{D} \times \vec{\Psi}. \quad (23)$$

A second primary constraint follows from the fact that the action has no dependence on  $d\Psi_0^\dagger/dt$ , which implies that the momentum conjugate to  $\Psi_0^\dagger$  vanishes identically,

$$P_{\Psi_0^\dagger} = 0. \quad (24)$$

Contracting  $\vec{V}$  with  $\vec{\sigma}$  and with  $g^{-1}\vec{D}$ , and using the covariant derivative relations of Eq. (A14), we get respectively

$$\begin{aligned} \vec{\sigma} \cdot \vec{V} &= 2i\theta + \chi, \\ g^{-1}\vec{D} \cdot \vec{V} &= i\omega + g^{-1}D_0\chi, \end{aligned} \quad (25)$$

with

$$\begin{aligned} \theta &\equiv \vec{\sigma} \cdot \vec{D} \Psi_0 - D_0 \vec{\sigma} \cdot \vec{\Psi}, \\ \omega &\equiv \vec{\sigma} \cdot \vec{B} \Psi_0 - (\vec{B} + \vec{\sigma} \times \vec{E}) \cdot \vec{\Psi}. \end{aligned} \quad (26)$$

Since the Euler-Lagrange equations imply that  $\vec{V}$  and  $\chi$  vanish for all times, we learn that  $\theta$  and  $\omega$  vanish also for all times. Since  $\theta$  involves a time derivative, its vanishing is just one component of the equation of motion for  $\Psi_\mu$ . But  $\omega$  involves no time derivatives, so it is a secondary constraint that relates  $\Psi_0$  to  $\vec{\Psi}$  (given in four-component form in Eq. (6)). For each of the above equations, there is a corresponding relation for the adjoint quantity.

The equation of motion  $\vec{V} = 0$  can be written in a simpler form by using the identities of Eqs. (A10) and (A11) as follows. Using Eq. (A10) to simplify  $0 = \vec{\sigma} \times \vec{V} - i\vec{V}$ , we get an equation for  $D_0\vec{\Psi}$ ,

$$D_0\vec{\Psi} = \vec{D}\Psi_0 + \frac{1}{2}[-\vec{\sigma} \times (\vec{D} \times \vec{\Psi}) + i\vec{D} \times \vec{\Psi}] \quad . \quad (27)$$

A further simplification can be achieved by incorporating the primary constraint  $\chi = 0$ , through applying Eq. (A11) to  $\vec{A} = \vec{D} \times \vec{\Psi}$ ,

$$0 = \vec{\sigma} \cdot \chi = \vec{\sigma} \cdot \vec{\sigma} \cdot (\vec{D} \times \vec{\Psi}) = \vec{D} \times \vec{\Psi} - i\vec{\sigma} \times (\vec{D} \times \vec{\Psi}) \quad . \quad (28)$$

Using this to replace the first term in square brackets in Eq. (27) we get the alternative form of the equation of motion, valid when the constraint  $\chi = 0$  is satisfied,

$$D_0\vec{\Psi} = \vec{D}\Psi_0 + i\vec{D} \times \vec{\Psi} \quad . \quad (29)$$

Writing the gauge field interaction terms in Eq. (21) in the form

$$S_{\text{int}}(\Psi_\mu) = \frac{g}{2} \int d^4x (A_0^B J^{B0} + \vec{A}^B \cdot \vec{J}^B) \quad , \quad (30)$$

we find the left chiral contribution to the current of Eq. (12) in the form

$$\begin{aligned} J^{A0} &= -\vec{\Psi}^\dagger t_A \cdot \vec{\sigma} \times \vec{\Psi} \quad , \\ \vec{J}^A &= \Psi_0^\dagger t_A \vec{\sigma} \times \vec{\Psi} + \vec{\Psi}^\dagger \times \vec{\sigma} t_A \Psi_0 - \vec{\Psi}^\dagger \times t_A \vec{\Psi} \quad . \end{aligned} \quad (31)$$

Replacing  $t_A$  by  $-i$ , we find the corresponding singlet current in the form

$$\begin{aligned} J^0 &= i\vec{\Psi}^\dagger \cdot \vec{\sigma} \times \vec{\Psi} \quad , \\ \vec{J} &= -i(\Psi_0^\dagger \vec{\sigma} \times \vec{\Psi} + \vec{\Psi}^\dagger \times \vec{\sigma} \Psi_0 - \vec{\Psi}^\dagger \times \vec{\Psi}) \quad . \end{aligned} \quad (32)$$

For the energy integral computed from the left chiral part of the the stress-energy tensor of Eq. (15), we find

$$H = - \int d^3x T_{RS}^{00} = -\frac{1}{2} \int d^3x \vec{\Psi}^\dagger \cdot \vec{D} \times \vec{\Psi} \quad . \quad (33)$$

To conclude this section, we verify that the action of Eq. (21) has a fermionic gauge invariance on the constraint surface  $\omega = 0$ ,  $\omega^\dagger = 0$ , as already seen in covariant form following Eq. (9). Letting  $\epsilon$  be a general space and time dependent two-component spinor, we introduce the fermionic gauge changes

$$\begin{aligned}\vec{\Psi} &\rightarrow \vec{\Psi} + \delta_G \vec{\Psi} \quad , \quad \delta_G \vec{\Psi} \equiv \vec{D}\epsilon \quad , \\ \Psi_0 &\rightarrow \Psi_0 + \delta_G \Psi_0 \quad , \quad \delta_G \Psi_0 \equiv D_0 \epsilon \quad ,\end{aligned}\tag{34}$$

and their adjoints, which are the left chiral form of the gauge change of Eq. (9). Substituting this into Eq. (21), integrating by parts where needed, and using Eqs. (A14) to simplify commutators of covariant derivatives, we find that Eq. (10) takes the two-component spinor form

$$\delta_G S(\Psi_\mu) = \frac{1}{2}ig \int d^4x (\omega^\dagger \epsilon - \epsilon^\dagger \omega) \quad .\tag{35}$$

Hence the action on the constraint surface  $\omega = \omega^\dagger = 0$  has a fermionic gauge invariance. Another gauge invariant, on the constraint surface  $\chi = \chi^\dagger = 0$ , is the fermion number, given by the space integral of the time component of the singlet current of Eq. (32),  $\int d^3x J^0$ , which has the gauge variation

$$\delta_G \int d^3x J^0 = -i \int d^3x (\epsilon^\dagger \chi + \chi^\dagger \epsilon) \quad .\tag{36}$$

However, neither the equation of motion, the constraints  $\chi$  and  $\omega$ , the non-Abelian “charge”  $\int d^3x J^{B0}$ , nor the integrated Hamiltonian  $H$  are gauge invariant in the interacting case. Using  $\delta_G$  to denote gauge variations, we have

$$\begin{aligned}\delta_G \vec{V} &= -ig(\vec{B} + \vec{\sigma} \times \vec{E})\epsilon \quad , \\ \delta_G \theta &= -ig\vec{\sigma} \cdot \vec{E}\epsilon \quad , \\ \delta_G \chi &= -ig\vec{\sigma} \cdot \vec{B}\epsilon \quad , \\ \delta_G \omega &= \vec{\sigma} \cdot \vec{B}D_0\epsilon - (\vec{B} + \vec{\sigma} \times \vec{E}) \cdot \vec{D}\epsilon \quad , \\ \delta_G \int d^3x J^{B0} &= g \int d^3x \left( \epsilon^\dagger [\vec{A}, t_B] \cdot \vec{\sigma} \times \vec{\Psi} + \vec{\Psi}^\dagger \times \vec{\sigma} \cdot [t_B, \vec{A}]\epsilon \right) \quad , \\ \delta_G H &= \frac{1}{2}ig \int d^3x (\vec{\Psi}^\dagger \cdot \vec{B}\epsilon - \epsilon^\dagger \vec{B} \cdot \vec{\Psi}) \quad .\end{aligned}\tag{37}$$

The last line shows that in the interacting case, there is no total energy integral that is invariant under the fermionic gauge transformation, much as there is no generally covariant gravitational

total energy. The *only* global fermionic gauge invariants are the action integral, and the fermion number integral.

To break the gauge invariance we must introduce an additional constraint, in the form

$$f(\vec{\Psi}) = 0 \quad , \quad (38)$$

with  $f$  a scalar function of its argument, such as  $f = \vec{D} \cdot \vec{\Psi}$  (a gauge covariant radiation gauge analog) or  $f = (\vec{B} + \vec{\sigma} \times \vec{E}) \cdot \vec{\Psi}$  (which by Eq. (26) corresponds, when  $\vec{\sigma} \cdot \vec{B}$  is invertible, to  $\Psi_0 = 0$ ). This constraint, together with the  $\chi$  constraint, leaves one independent two-component spinor of the original three in  $\vec{\Psi}$ , corresponding to the physical massless Rarita-Schwinger modes propagating in the gauge field background. We will limit ourselves to considering linear constraints of the general form

$$f = \vec{L} \cdot \vec{\Psi} \quad , \quad (39)$$

so that for gauge covariant radiation gauge we have  $\vec{L} = \vec{D}$ , and for  $\Psi_0 = 0$  gauge we have  $\vec{L} = \vec{B} + \vec{\sigma} \times \vec{E}$ . Both of these choices of  $\vec{L}$  play a special role in our analysis, so we shall examine them in more detail.

We consider first the gauge covariant radiation gauge. We note that since

$$\vec{\sigma} \cdot \vec{D} \vec{\sigma} \cdot \vec{\Psi} = \vec{D} \cdot \vec{\Psi} + i\chi, \quad (40)$$

the primary constraint  $\chi = 0$  implies that

$$\vec{\sigma} \cdot \vec{D} \vec{\sigma} \cdot \vec{\Psi} = \vec{D} \cdot \vec{\Psi} \quad . \quad (41)$$

Hence when  $\vec{\sigma} \cdot \vec{D}$  is invertible, which is expected in a perturbation expansion in the gauge coupling  $g$ , the covariant radiation gauge constraint  $\vec{D} \cdot \vec{\Psi} = 0$  implies that

$$\vec{\sigma} \cdot \vec{\Psi} = 0 \quad . \quad (42)$$

Conversely, Eqs. (40) and (41) show that  $\vec{D} \cdot \vec{\Psi} = 0$  and  $\vec{\sigma} \cdot \vec{\Psi} = 0$  together imply the primary constraint  $\chi = 0$ , and also  $\vec{\sigma} \cdot \vec{\Psi} = 0$  and  $\chi = 0$  imply  $\vec{D} \cdot \vec{\Psi} = 0$ .

We next note that on a given initial time slice, covariant radiation gauge is attainable. Under the gauge transformation of Eq. (34), we see that

$$\vec{D} \cdot \vec{\Psi} \rightarrow \vec{D} \cdot \vec{\Psi} + (\vec{D})^2 \epsilon \quad . \quad (43)$$

Hence when  $(\vec{D})^2$  is invertible, which we expect to be true in a perturbative sense, then we can invert  $(\vec{D})^2\epsilon = -\vec{D} \cdot \vec{\Psi}$ , to find a gauge function  $\epsilon$  that brings a general  $\vec{\Psi}$  to covariant radiation gauge. Since

$$(\vec{\sigma} \cdot \vec{D})^2 = (\vec{D})^2 + g\vec{\sigma} \cdot \vec{B} \quad , \quad (44)$$

the conditions for  $\vec{\sigma} \cdot \vec{D}$  to be invertible, and for  $(\vec{D})^2$  to be invertible, are related. For generic non-Abelian gauge fields both of these operators should be invertible, but there will be isolated gauge field configurations for which  $\vec{\sigma} \cdot \vec{D}$  has zeros.

However, although covariant radiation gauge can be imposed on any time slice, it is not preserved by the equation of motion for  $\vec{\Psi}$ . To see this, let us consider the simplified case in which the gauge potential is specialized to  $A_0 = 0$  and  $\partial_0 \vec{A} = 0$ , so that only a static  $\vec{B}$  field is present. From Eq. (29) we have

$$\partial_0(\vec{D} \cdot \vec{\Psi}) = (\vec{D})^2\Psi_0 + g\vec{B} \cdot \vec{\Psi} = ((\vec{D})^2(\vec{\sigma} \cdot \vec{B})^{-1} + g)\vec{B} \cdot \vec{\Psi} \neq 0 \quad . \quad (45)$$

Hence at each infinitesimal time step, we must make a further infinitesimal gauge transformation to maintain the gauge covariant radiation gauge condition. This means that we *cannot* use the  $\theta = 0$  time development equation

$$\vec{\sigma} \cdot \vec{D}\Psi_0 = D_0\vec{\sigma} \cdot \vec{\Psi} \quad , \quad (46)$$

together with Eq. (42), and the assumption that  $\vec{\sigma} \cdot \vec{D}$  is invertible, to conclude that covariant radiation gauge also implies that  $\Psi_0 = 0$ .

We can, however, impose  $\Psi_0 = 0$  as an alternative gauge condition. Under the gauge transformation of Eq. (34),  $\Psi_0$  can be reduced to zero by taking

$$\begin{aligned} \epsilon &= -V(t, \vec{x})^{-1} \int_0^t du V(u, \vec{x}) \Psi_0(u, \vec{x}) \quad , \\ V(t, \vec{x}) &= \exp \left( g \int_0^t du A_0(u, \vec{x}) \right) \quad . \end{aligned} \quad (47)$$

Once  $\Psi_0$  has been gauged to zero, we can conclude from Eq. (26) that  $(\vec{B} + \vec{\sigma} \times \vec{E}) \cdot \vec{\Psi} = 0$ . Just as we found for covariant radiation gauge,  $\Psi_0 = 0$  gauge is not preserved by the equation of motion for  $\vec{\Psi}$ . To see this, let us again consider the simplified case in which the gauge potential is specialized to  $A_0 = 0$  and  $\partial_0 \vec{A} = 0$ , so that only a static  $\vec{B}$  field is present. From Eq. (29) we have

$$\partial_0(\vec{B} \cdot \vec{\Psi}) = i\vec{B} \cdot \vec{D} \times \vec{\Psi} \neq 0 \quad . \quad (48)$$

So again, at each infinitesimal time step, we must make a further infinitesimal gauge transformation to maintain the gauge condition  $\Psi_0 = 0$ .

To summarize this discussion of gauge fixing, we see that when the gauge fields are nonzero, we can have either  $\vec{D} \cdot \vec{\Psi} = 0$  or  $\Psi_0 = 0$ , but not both. In the free field case, these two conditions can be imposed simultaneously and are preserved by the time evolution of  $\vec{\Psi}$ , so we recover the constraints  $\vec{\nabla} \cdot \vec{\psi} = 0$ ,  $\vec{\sigma} \cdot \vec{\psi} = 0$ , and  $\psi_0 = 0$  used in the discussion of [8] for the free Rarita-Schwinger case,

## V. PROPAGATION OF A RARITA-SCHWINGER FIELD IN AN EXTERNAL ABELIAN GAUGE FIELD: ABSENCE OF SUPERLUMINAL PROPAGATION

We specialize now to the case of a Rarita-Schwinger spinor propagating in an external Abelian gauge field, as studied by Velo and Zwanziger [4]. For an Abelian gauge field,

$$\frac{1}{\vec{\sigma} \cdot \vec{B}} = \frac{\vec{\sigma} \cdot \vec{B}}{(\vec{B})^2} \quad , \quad (49)$$

and so  $\vec{\sigma} \cdot \vec{B}$  is invertible as long as  $(\vec{B})^2 > 0$ , which we assume. Provided the Lorentz invariant expression  $(\vec{B})^2 - (\vec{E})^2$  is positive,  $(\vec{B})^2$  will be positive in any Lorentz frame. In discussing undamped wave propagation we will not use the inequality  $(\vec{B})^2 - (\vec{E})^2 > 0$ , but in treating damped longitudinal mode propagation in Appendix B, we will assume that  $(\vec{E})^2/(\vec{B})^2$  is small, as motivated by the fact that when  $(\vec{E})^2$  is of order  $(\vec{B})^2$  the vacuum is highly unstable against pair creation. (Strictly speaking, the vacuum is stable against pair production only when  $\vec{E} \cdot \vec{B} = 0$  and  $(\vec{B})^2 - (\vec{E})^2 > 0$ , that is, when there is a Lorentz frame in which the Abelian field has vanishing  $\vec{E}$  [12].)

Given that  $(\vec{B})^2 > 0$ , we can solve the constraint  $\omega = 0$  of Eq. (26) for  $\Psi_0$ , giving

$$\Psi_0 = \frac{\vec{Q} \cdot \vec{\Psi}}{(\vec{B})^2} \quad , \quad (50)$$

where we have defined

$$\vec{Q} \equiv \vec{\sigma} \cdot \vec{B}(\vec{B} + \vec{\sigma} \times \vec{E}) = \vec{B} \times \vec{E} + \vec{B}\vec{\sigma} \cdot (\vec{B} + i\vec{E}) - i\vec{B} \cdot \vec{E}\vec{\sigma} \quad . \quad (51)$$

Substituting the solution for  $\Psi_0$  into Eq. (29), we get an equation of motion for  $\vec{\Psi}$  by itself,

$$D_0 \vec{\Psi} = \vec{D} \frac{\vec{Q} \cdot \vec{\Psi}}{(\vec{B})^2} + i\vec{D} \times \vec{\Psi} \quad . \quad (52)$$

To determine the wave propagation velocity in the neighborhood of a point  $\vec{x}_*$ , we need to calculate the equation for the wavefronts, or characteristics, at that point. Writing the first order Eq. (52) in the form

$$\partial_0 \vec{\Psi} = \vec{\nabla} \frac{\vec{Q}_* \cdot \vec{\Psi}}{(\vec{B}_*)^2} + i\vec{\nabla} \times \vec{\Psi} + \vec{\Delta}[\vec{\Psi}, \vec{x}_*, \vec{x}] \quad , \quad (53)$$

with  $\vec{B}_*$  and  $\vec{Q}_*$  the values of the respective quantities at  $\vec{x}_*$ , we see that  $\vec{\Delta}[\vec{\Psi}, \vec{x}_*, \vec{x}]$  involves no first derivatives of  $\vec{\Psi}$  at  $\vec{x}_*$ , and so is not needed [13], [14] for determining the wavefronts of Eq. (29). The reason is that when taking an infinitesimal line integral of Eq. (53), according to

$$\lim_{\delta \rightarrow 0} \int_{-\delta}^{\delta} [\partial_0 \vec{\Psi} = \dots] \quad , \quad (54)$$

discontinuities across wavefronts contribute through the first derivative terms, but when the external fields are smooth the term  $\vec{\Delta}[\vec{\Psi}, \vec{x}_*, \vec{x}]$  makes a vanishing contribution as  $\delta \rightarrow 0$ . Dropping  $\vec{\Delta}$ , and multiplying through by  $(\vec{B}_*)^2$ , we get the equation determining the wavefronts in the form

$$(\vec{B}_*)^2 \partial_0 \vec{\Psi} = \vec{\nabla} \vec{Q}_* \cdot \vec{\Psi} + i(\vec{B}_*)^2 \vec{\nabla} \times \vec{\Psi} \quad . \quad (55)$$

By similar reasoning, the constraint  $\chi$  can be simplified, for purposes of determining the wavefronts, by replacing  $\vec{D}$  by  $\vec{\nabla}$ , giving

$$0 = \vec{\sigma} \cdot \vec{\nabla} \times \vec{\Psi} \quad , \quad (56)$$

and the covariant radiation gauge condition similarly simplifies to

$$0 = \vec{\nabla} \cdot \vec{\Psi} \quad . \quad (57)$$

Since these are now linear equations with constant coefficients, the solutions are plane waves, and without loss of generality we can take the negative  $z = x_3$  axis as the direction of wave propagation. So making the Ansatz

$$\vec{\Psi} = \vec{C} \exp(i\Omega t + iKz) \quad , \quad (58)$$

Eq. (55) for the wavefronts or characteristics takes the form

$$0 = \vec{F} \equiv (\vec{B}_*)^2 \Omega \vec{C} - K \hat{z} \vec{Q}_* \cdot \vec{C} - i(\vec{B}_*)^2 K \hat{z} \times \vec{C} \quad , \quad (59)$$

with  $\hat{z}$  a unit vector along the  $z$  axis. Similarly, the constraint Eq. (56) becomes an admissability condition on  $\vec{C}$ ,

$$0 = \vec{\sigma} \cdot \hat{z} \times \vec{C} \quad , \quad (60)$$

and the gauge fixing condition  $\vec{\nabla} \cdot \vec{\Psi} = 0$  imposes the further condition on  $\vec{C}$

$$0 = \hat{z} \cdot \vec{C} \quad , \quad (61)$$

but for the moment we will analyze the solutions of Eq. (59) without assuming this additional constraint.



Writing  $F_m$  as a matrix times  $C_n$  (and dropping the subscripts  $*$ , which are implicit from here on) we have

$$\begin{aligned} F_m &= N_{mn} C_n \quad , \\ N_{mn} &= (\vec{B})^2 \Omega \delta_{mn} - K \delta_{m3} Q_n - i(\vec{B})^2 K \epsilon_{m3n} \quad . \end{aligned} \quad (62)$$

The equation for the characteristics is now

$$\det(N) = 0 \quad , \quad (63)$$

since this is the condition for Eq. (59) to have a solution with nonzero  $\vec{C}$ . However, since evaluation of the determinant shows that it factorizes into blocks that determine  $C_{1,2}$  and a block that determines  $C_3$ , a simpler way to proceed is to work directly from the equations  $F_m = 0$ , which decouple in a corresponding way. Calculating from Eq. (59), we find

$$\begin{aligned} 0 &= F_1^{\uparrow,\downarrow} = (\vec{B})^2 (\Omega C_1^{\uparrow,\downarrow} + iK C_2^{\uparrow,\downarrow}) \quad , \\ 0 &= F_2^{\uparrow,\downarrow} = (\vec{B})^2 (\Omega C_2^{\uparrow,\downarrow} - iK C_1^{\uparrow,\downarrow}) \quad , \\ 0 &= F_3^{\uparrow,\downarrow} = (\vec{B})^2 \Omega C_3^{\uparrow,\downarrow} - K(\vec{Q} \cdot \vec{C})^{\uparrow,\downarrow} \quad , \end{aligned} \quad (64)$$

where  $\uparrow, \downarrow$  indicate the up and down spinor components, labeled in Eq. (20) by  $\alpha = 1, 2$ . Similarly, the constraint Eq. (60) becomes  $0 = -\sigma_1 C_2 + \sigma_2 C_1$ , that is

$$\begin{aligned} C_2^{\uparrow} &= iC_1^{\uparrow} \quad , \\ C_2^{\downarrow} &= -iC_1^{\downarrow} \quad , \end{aligned} \quad (65)$$

with no corresponding condition on  $C_3^{\uparrow,\downarrow}$ .

The first two lines of Eq. (64) together with Eq. (65) have the solution

$$\begin{aligned} C_1^{\uparrow} &= C \quad , \quad C_2^{\uparrow} = iC \quad , \quad \Omega = K \quad , \\ C_1^{\downarrow} &= C \quad , \quad C_2^{\downarrow} = -iC \quad , \quad \Omega = -K \quad , \end{aligned} \quad (66)$$

with  $C$  arbitrary, corresponding to waves with velocity of magnitude  $|\Omega/K| = 1$ . The effect of a gauge change  $\vec{\Psi} \rightarrow \vec{\Psi} + \vec{D}\epsilon$ ,  $\epsilon = (E/(iK)) \exp(i\Omega t + iKz)$  on the characteristics is to shift

$C_3^{\uparrow,\downarrow} \rightarrow C_3^{\uparrow,\downarrow} + E^{\uparrow,\downarrow}$ , thus replacing the third line of Eq. (64) by the two-component spinor equation  $((\vec{B})^2\Omega - KQ_3)(C_3 + E) - K(Q_1C_1 + Q_2C_2) = 0$ . If we choose  $E$  to cancel the term  $K(Q_1C_1 + Q_2C_2)$ , that is,

$$E = \frac{K(Q_1C_1 + Q_2C_2)}{(\vec{B})^2\Omega - KQ_3} , \quad (67)$$

then this equation is solved by  $C_3 = 0$  (corresponding to enforcing the covariant radiation gauge conditions  $\vec{\nabla} \cdot \vec{\Psi} = \vec{\sigma} \cdot \vec{\Psi} = 0$ ), and the demonstration that there is no superluminal propagation is complete. In Appendix B, we continue this discussion without assuming a specific gauge condition, and show that there are still no propagating modes with superluminal velocities. That is, the purely longitudinal mode with  $C_1 = C_2 = 0, C_3 \neq 0$ , which is eliminated by the gauge fixing condition, does not propagate superluminally.

## VI. CANONICAL MOMENTA, CLASSICAL BRACKETS, AND GAUGE GENERATORS

We return now to the general non-Abelian gauge field case, and introduce the canonical momentum conjugate to  $\vec{\Psi}$ , defined by

$$\vec{P} = \frac{\partial^L S}{\partial(\partial_0 \vec{\Psi})} = \frac{1}{2} \vec{\Psi}^\dagger \times \vec{\sigma} , \quad (68)$$

which can be solved for  $\vec{\Psi}^\dagger$  using the final line of Eq. (A11),

$$\vec{\Psi}^\dagger = i\vec{P} - \vec{P} \times \vec{\sigma} . \quad (69)$$

We will use Eq. (69) when computing classical brackets involving  $\vec{\Psi}^\dagger$  using the formula of Eq. (A17). Eq. (68) can be written as an explicit matrix relation for the six components of  $\vec{P}$  and  $\vec{\Psi}^\dagger$ ,

$$\begin{pmatrix} P_1^\uparrow \\ P_1^\downarrow \\ P_2^\uparrow \\ P_2^\downarrow \\ P_3^\uparrow \\ P_3^\downarrow \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & -i \\ 0 & 0 & 0 & -1 & i & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & i & 0 & -1 & 0 & 0 \\ -i & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Psi_1^{\uparrow\uparrow} \\ \Psi_1^{\uparrow\downarrow} \\ \Psi_2^{\uparrow\uparrow} \\ \Psi_2^{\uparrow\downarrow} \\ \Psi_3^{\uparrow\uparrow} \\ \Psi_3^{\uparrow\downarrow} \end{pmatrix} , \quad (70)$$

showing that they are related by an anti-self-adjoint matrix with determinant  $-1/16$ .

The four constraints introduced in Sec. 4 are

$$\begin{aligned}
\phi_1 &= P_{\Psi_0^\dagger} \quad , \\
\phi_2 &= (\vec{\sigma} \cdot \vec{B})^{-1} \omega = \Psi_0 - (\vec{\sigma} \cdot \vec{B})^{-1} (\vec{B} + \vec{\sigma} \times \vec{E}) \cdot \vec{\Psi} \quad , \\
\phi_3 &= \chi = \vec{\sigma} \cdot \vec{D} \times \vec{\Psi} \quad , \\
\phi_4 &= \vec{L} \cdot \vec{\Psi} \quad .
\end{aligned} \tag{71}$$

In writing these we are assuming that  $\vec{\sigma} \cdot \vec{B}$  is invertible in the non-Abelian case. We are writing the gauge fixing condition as a general linear gauge fixing constraint  $\vec{L} \cdot \vec{\Psi}$ , so as to keep track of which terms in the final answers arise from gauge fixing, which is not evident if we specialize by replacing  $\vec{L}$  by  $\vec{D}$  or  $\vec{B} + \vec{\sigma} \times \vec{E}$ . The constraints of Eq. (71), including the gauge fixing constraint  $\phi_4$ , are all first class in the Dirac classification, since they have vanishing mutual classical brackets. This is a consequence of the fact that starting with a constraint depending on  $\vec{\Psi}$  but not on  $\vec{\Psi}^\dagger$ , and taking an arbitrary number of time derivatives, one still has a constraint depending only on  $\vec{\Psi}$ .

To preserve the adjoint properties of the Rarita-Schwinger equation, for each of these four constraints we must impose a corresponding adjoint constraint. Using Eq. (69) to express  $\vec{\Psi}^\dagger$  in terms of  $\vec{P}$ , we write these as

$$\begin{aligned}
\chi_1 &= (P_{\Psi_0^\dagger})^\dagger = -P_{\Psi_0} \quad , \\
\chi_2 &= \omega^\dagger (\vec{\sigma} \cdot \vec{B})^{-1} = \Psi_0^\dagger - \vec{P} \cdot [i(\vec{B} + \vec{\sigma} \times \vec{E}) - \vec{\sigma} \times (\vec{B} + \vec{\sigma} \times \vec{E})] (\vec{\sigma} \cdot \vec{B})^{-1} \quad , \\
\chi_3 &= \chi^\dagger = 2\vec{P} \cdot \overleftarrow{D} \quad , \\
\chi_4 &= \vec{\Psi}^\dagger \cdot \overleftarrow{L} = \vec{P} \cdot (i\overleftarrow{L} - \vec{\sigma} \times \overleftarrow{L}) \quad .
\end{aligned} \tag{72}$$

(The reason for the minus sign in the definition  $P_{\Psi_0^\dagger} = -P_{\Psi_0}^\dagger$  will be given in Sec. 8 where we discuss the Hamiltonian form of the equations.) The constraints  $\phi_a$  are implicitly  $2n$  component column vectors, and the adjoint constraints  $\chi_a$  are implicitly  $2n$  component row vectors, with  $2n$  arising from the product of a factor of 2 for the two implicit spinor indices, and a factor of  $n$  for the  $n$  implicit  $SU(n)$  internal symmetry indices.

When  $\vec{L} = \vec{D}$ , we see that  $\phi_4$  becomes  $\phi_4 = \vec{D} \cdot \vec{\Psi}$ , and  $\chi_4$  becomes  $\chi_4 = i\vec{P} \cdot \overleftarrow{D} - \vec{P} \cdot \vec{\sigma} \times \overleftarrow{D} = (i/2)\chi_3 - \vec{P} \cdot \vec{\sigma} \times \overleftarrow{D}$ . So a special feature of covariant radiation gauge, which will be exploited later, is that the constraints  $\phi_3, \phi_4$  are contractions of  $\vec{\sigma} \times \vec{D}$  and  $\vec{D}$  with  $\vec{\Psi}$ , and the constraints  $\chi_3, \chi_4$

are contractions of linear combinations of the duals  $\overleftarrow{D}$  and  $\vec{\sigma} \times \overleftarrow{D}$  with  $\vec{P}$ . That is, in covariant radiation gauge the constraint spaces selected by  $\chi_3, \chi_4$  and  $\phi_3, \phi_4$  are similar.

We can now compute the classical brackets of the constraints. We see that the brackets of the  $\phi$ s and  $\chi$ s vanish among themselves,

$$\begin{aligned} [\phi_a, \phi_b]_C &= 0 \quad , \\ [\chi_a, \chi_b]_C &= 0 \quad , \\ a, b &= 1, \dots, 4 \quad . \end{aligned} \tag{73}$$

On the other hand, the brackets of the  $\phi$ s with the  $\chi$ s give a nontrivial matrix of brackets  $M$ , which has a nonvanishing determinant,

$$\begin{aligned} M_{ab}(\vec{x}, \vec{y}) &\equiv [\phi_a(\vec{x}), \chi_b(\vec{y})]_C \neq 0 \quad , \\ \det M &\neq 0 \quad . \end{aligned} \tag{74}$$

Thus, in terms of the Dirac classification, the original first class constraints  $\phi_a$  have become second class, not from adding new constraints that follow from differentiation with respect to time or from imposing gauge fixing conditions, but rather from adjoining the adjoint set of constraints. This is a feature of the Rarita-Schwinger constrained fermion system that has no analog in the familiar constrained boson systems such as gauge fields.

Evaluating the brackets shows that  $M$  has the general form

$$M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & \mathcal{U} & \mathcal{S} & \mathcal{T} \\ 0 & \mathcal{V} & \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{W} & \mathcal{C} & \mathcal{D} \end{pmatrix} \quad , \tag{75}$$

where in the  $SU(n)$  gauge field case, each entry in  $M$  is a  $2n \times 2n$  matrix (corresponding to the fact that  $\phi_a$  is implicitly a  $2n$  component column vector, and  $\chi_b$  is implicitly a  $2n$  component row vector). Evaluating  $\det M$  by a cofactor expansion with respect to the elements of the two unit

matrices  $\pm 1$ , we see that the submatrices  $\mathcal{U}, \mathcal{S}, \mathcal{T}, \mathcal{V}, \mathcal{W}$  do not contribute, and we have

$$\det M = \det N$$

$$N = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} .$$
(76)

So we need to only evaluate the brackets  $M_{33} = \mathcal{A}$ ,  $M_{34} = \mathcal{B}$ ,  $M_{43} = \mathcal{C}$ ,  $M_{44} = \mathcal{D}$ , giving

$$\begin{aligned} \mathcal{A} &= -2ig\vec{\sigma} \cdot \vec{B}(\vec{x})\delta^3(\vec{x} - \vec{y}) \quad , \\ \mathcal{B} &= -2\vec{D}_{\vec{x}} \cdot \vec{L}_{\vec{x}}\delta^3(\vec{x} - \vec{y}) \quad , \\ \mathcal{C} &= 2\vec{L}_{\vec{x}} \cdot \vec{D}_{\vec{x}}\delta^3(\vec{x} - \vec{y}) \quad , \\ \mathcal{D} &= (i(\vec{L}_{\vec{x}})^2 + \vec{\sigma} \cdot (\vec{L}_{\vec{x}} \times \vec{L}_{\vec{x}}))\delta^3(\vec{x} - \vec{y}) \quad . \end{aligned}$$
(77)

When  $\vec{L} = \vec{D}$ , these become

$$\begin{aligned} \mathcal{A} &= -2ig\vec{\sigma} \cdot \vec{B}(\vec{x})\delta^3(\vec{x} - \vec{y}) \quad , \\ \mathcal{B} &= -2(\vec{D}_{\vec{x}})^2\delta^3(\vec{x} - \vec{y}) \quad , \\ \mathcal{C} &= 2(\vec{D}_{\vec{x}})^2\delta^3(\vec{x} - \vec{y}) \quad , \\ \mathcal{D} &= i((\vec{D}_{\vec{x}})^2 - g\vec{\sigma} \cdot \vec{B}(\vec{x}))\delta^3(\vec{x} - \vec{y}) \quad . \end{aligned}$$
(78)

Reflecting the fact that the  $\phi_a$  and  $\chi_a$  are adjoints of one another, together with the fact that the matrix relating  $\vec{\Psi}^\dagger$  to  $\vec{P}$  is anti-self-adjoint (see Eq. (70)), these matrix elements obey the adjoint relations

$$M_{ab}(\vec{x}, \vec{y})^\dagger = -M_{ba}(\vec{y}, \vec{x}) \quad .$$
(79)

Applications of these bracket and determinant calculations will be made in the next two sections.

To conclude this section, we note that the constraints  $\chi, \chi^\dagger, P_{\Psi_0}, P_{\Psi_0^\dagger}$  play the role of gauge transformation generators. For example, we have (with common time argument  $t$  suppressed)

$$\begin{aligned} \left[ \int d^3x \frac{1}{2} \chi^\dagger(\vec{x}) \epsilon(\vec{x}), \vec{\Psi}(\vec{y}) \right]_C &= \vec{D}_{\vec{y}} \epsilon(\vec{y}) \quad , \\ \left[ - \int d^3x P_{\Psi_0}(\vec{x}) D_0 \epsilon(\vec{x}), \Psi_0(\vec{y}) \right]_C &= D_{0\vec{y}} \epsilon(\vec{y}) \quad . \end{aligned}$$
(80)

So the fermionic gauge transformation is a canonical transformation.

## VII. PATH INTEGRAL QUANTIZATION IN $\Psi_0 = 0$ GAUGE

In studying quantization, we will specialize to the case where the external gauge potentials, and hence  $\vec{D}$ , are time independent, since the simplest discussions of constrained systems assume time-independent constraints. This assumption can be dropped when the gauge field is quantized along with the Rarita-Schwinger field, but a more complex system of constraints and constraint brackets will then appear; we defer this extension to a future investigation.

When the constraints are time independent, the classical brackets of Eqs. (73) and (74) have the form needed to apply the Faddeev-Popov [15] method for path integral quantization. (This has been applied in the free Rarita-Schwinger case by Das and Freedman [8] and by Senjanović [16].) The general formula of [15] for the in to out  $S$  matrix element (up to a constant proportionality factor) reads

$$\begin{aligned} \langle \text{out} | S | \text{in} \rangle &\propto \int \exp(iS(q, p)) \prod_t d\mu(q(t), p(t)) \quad , \\ d\mu(q, p) &= \prod_a \delta(\chi_a) \delta(\phi_a) (\det[\phi_a, \chi_b])^\xi \prod_i dp_i dq_i \quad , \end{aligned} \quad (81)$$

where  $\xi = 1$  when all canonical variables are bosonic, and  $\xi = -1$  in our case in which all canonical variables are fermionic, or Grassmann odd. In applying this formula, we note that since the action  $S$  of Eq. (21) and the bracket matrix  $M$  of Eqs. (74)-(77) are independent of  $P_{\Psi_0}$  and  $P_{\Psi_0^\dagger}$ , we can immediately integrate out the delta functions in these two constraints. Also, since the canonical momentum  $\vec{P}$  is related to  $\vec{\Psi}^\dagger$  by the constant numerical transformation of Eqs. (69) and (70), we can take  $\vec{\Psi}^\dagger$  as the integration variable instead of  $\vec{P}$ , up to an overall proportionality constant. So we have the formula

$$\begin{aligned} \langle \text{out} | S | \text{in} \rangle &\propto \int \exp\left(i\frac{1}{2} \int d^4x [-\Psi_0^\dagger \vec{\sigma} \cdot \vec{D} \times \vec{\Psi} + \vec{\Psi}^\dagger \cdot \vec{\sigma} \times \vec{D} \Psi_0 + \vec{\Psi}^\dagger \cdot \vec{D} \times \vec{\Psi} - \vec{\Psi}^\dagger \cdot \vec{\sigma} \times D_0 \vec{\Psi}]\right) \\ &\times \prod_{t, \vec{x}} d\mu(\Psi_0, \Psi_0^\dagger, \vec{\Psi}, \vec{\Psi}^\dagger) \quad , \end{aligned} \quad (82)$$

with

$$d\mu(\Psi_0, \Psi_0^\dagger, \vec{\Psi}, \vec{\Psi}^\dagger) = \left( \prod_{c=2}^4 \delta(\chi_c) \delta(\phi_c) \right) (\det[\phi_a, \chi_b])^{-1} d\Psi_0 d\Psi_0^\dagger d\vec{\Psi} d\vec{\Psi}^\dagger \quad , \quad (83)$$

with  $d\Psi_0$  and  $d\Psi_0^\dagger$  each a product over the spinor components, and  $d\vec{\Psi}$  and  $d\vec{\Psi}^\dagger$  each a product over the spinor-vector components.

As our next step, we can carry out the integrations over  $\Psi_0$  and  $\Psi_0^\dagger$ , using the delta functions  $\delta(\phi_2)$  and  $\delta(\chi_2)$ . This leaves the formula

$$\begin{aligned} \langle \text{out} | S | \text{in} \rangle &\propto \int \exp \left( i \frac{1}{2} \int d^4 x \left[ -\vec{\Psi}^\dagger \cdot (\vec{B} + \vec{\sigma} \times \vec{E}) (\vec{\sigma} \cdot \vec{B})^{-1} \vec{\sigma} \cdot \vec{D} \times \vec{\Psi} \right. \right. \\ &\quad \left. \left. + \vec{\Psi}^\dagger \cdot \vec{\sigma} \times \vec{D} (\vec{\sigma} \cdot \vec{B})^{-1} (\vec{B} + \vec{\sigma} \times \vec{E}) \cdot \vec{\Psi} + \vec{\Psi}^\dagger \cdot \vec{D} \times \vec{\Psi} - \vec{\Psi}^\dagger \cdot \vec{\sigma} \times D_0 \vec{\Psi} \right] \right) \\ &\quad \times \prod_{t, \vec{x}} d\mu(\vec{\Psi}, \vec{\Psi}^\dagger) \quad , \end{aligned} \tag{84}$$

with

$$d\mu(\vec{\Psi}, \vec{\Psi}^\dagger) = \left( \prod_{c=3}^4 \delta(\chi_c) \delta(\phi_c) \right) (\det[\phi_a, \chi_b])^{-1} d\vec{\Psi} d\vec{\Psi}^\dagger \quad , \tag{85}$$

so that the only remaining constraints  $\phi_{3,4}$ ,  $\chi_{3,4}$  are the ones used in constructing the determinant  $\det[\phi_a, \chi_b]$ .

We now confront a dilemma: If we impose the constraint  $\chi = 0$  coming from  $\delta(\chi)$ , the first term in the exponent drops out, and we are left with

$$\begin{aligned} \langle \text{out} | S | \text{in} \rangle &\propto \int \exp \left( i \frac{1}{2} \int d^4 x \vec{\Psi}^\dagger \cdot [\vec{\sigma} \times \vec{D} (\vec{\sigma} \cdot \vec{B})^{-1} (\vec{B} + \vec{\sigma} \times \vec{E}) \cdot \vec{\Psi} + \vec{D} \times \vec{\Psi} - \vec{\sigma} \times D_0 \vec{\Psi}] \right) \\ &\quad \times \prod_{t, \vec{x}} d\mu(\vec{\Psi}, \vec{\Psi}^\dagger) \quad , \end{aligned} \tag{86}$$

in which the coefficient of  $\vec{\Psi}^\dagger$  in the exponent is  $-\vec{V}$ , and so at the stationary phase point of the exponent with respect to variations in  $\vec{\Psi}^\dagger$  we get the correct equation of motion. On the other hand, if we integrate the first term in the exponent of Eq. (86) by parts, it becomes  $-(i/2) \int d^4 x \chi^\dagger (\vec{\sigma} \cdot \vec{B})^{-1} (\vec{B} + \vec{\sigma} \times \vec{E}) \cdot \vec{\Psi}$ , which is set to zero by the constraint  $\chi^\dagger = 0$  coming from  $\delta(\chi^\dagger)$ ! Are we not allowed to integrate by parts here because the factor  $(\vec{\sigma} \cdot \vec{B})^{-1} (\vec{B} + \vec{\sigma} \times \vec{E})$  is not continuously differentiable? This dilemma is avoided by working in  $\Psi_0 = 0$  gauge, since as we have seen, then  $(\vec{B} + \vec{\sigma} \times \vec{E}) \cdot \vec{\Psi} = 0$  and the troublesome term is absent from both the path integral and the equation of motion. In  $\Psi_0 = 0$  gauge, we end up with the elegant formula

$$\begin{aligned} \langle \text{out} | S | \text{in} \rangle &\propto \int \exp \left( i \frac{1}{2} \int d^4 x \vec{\Psi}^\dagger \cdot [\vec{D} \times \vec{\Psi} - \vec{\sigma} \times D_0 \vec{\Psi}] \right) \\ &\quad \times \prod_{t, \vec{x}} d\mu(\vec{\Psi}, \vec{\Psi}^\dagger) \quad . \end{aligned} \tag{87}$$

In using this formula, the customary procedure [17] would be to put the bracket matrix that is the argument of the determinant back into the exponent by introducing bosonic ghost fields  $\phi_G$ . But since for  $\Psi_0 = 0$  gauge  $\vec{L} = \vec{B} + \vec{\sigma} \times \vec{E}$ ,  $\phi_G^\dagger N \phi_G$  is only linear in derivatives of  $\phi_G$ , so one would not get ghost propagators of the standard bosonic form.

### VIII. HAMILTONIAN FORM OF THE EQUATIONS AND THE DIRAC BRACKET

An alternative way to quantize is to transform the Lagrangian equations to Hamiltonian form, and to take the constraints into account by replacing the classical brackets by Dirac brackets. In carrying this out, we will simplify the formulas by making the gauge choice  $A_0 = 0$  for the non-Abelian gauge fields. This gauge choice is always attainable, and leaves a residual non-Abelian gauge invariance  $\vec{A} \rightarrow \vec{A} + \vec{D}\Lambda(\vec{x})$ , with the gauge parameter  $\Lambda$  time independent. The Hamiltonian will then be covariant with respect to this restricted gauge transformation. For the moment, in discussing the canonical Hamiltonian and bracket formalism, we will allow  $\vec{A}$  to be time dependent, so that  $\vec{E} \neq 0$ . But when we turn to the Dirac bracket construction corresponding to a constrained Hamiltonian, we will assume a time-independent  $\vec{A}$  as in the path integral discussion, corresponding in  $A_0 = 0$  gauge to  $\vec{E} = 0$ . (If we carry along the  $A_0$  term in the formulas, which we have done as a check, then time-independent fields would not require  $\vec{E} = 0$ . So this specialization can be avoided at the price of somewhat lengthier equations.)

From the action  $S(\Psi_\mu) = \int dt L(\Psi_\mu)$  of Eq. (21) and the canonical momentum of Eq. (68), we find the canonical Hamiltonian to be

$$\begin{aligned} H &= \int d^3x \partial_0 \vec{\Psi} \cdot \vec{P} - L \\ &= -\frac{1}{2} \int d^3x [-\Psi_0^\dagger \vec{\sigma} \cdot \vec{D} \times \vec{\Psi} + \vec{\Psi}^\dagger \cdot \vec{\sigma} \times \vec{D} \Psi_0 + \vec{\Psi}^\dagger \cdot \vec{D} \times \vec{\Psi}] \\ &= -\frac{1}{2} \int d^3x [-\Psi_0^\dagger \vec{\sigma} \cdot \vec{D} \times \vec{\Psi} + (i\vec{P} - \vec{P} \times \vec{\sigma}) \cdot (\vec{\sigma} \times \vec{D} \Psi_0 + \vec{D} \times \vec{\Psi})] \quad , \end{aligned} \tag{88}$$

where in the final line we have used Eq. (69) to express  $\vec{\Psi}^\dagger$  in terms of  $\vec{P}$ .

We can now compute the classical brackets of various quantities with  $H$ . From

$$\begin{aligned} \frac{d\vec{\Psi}}{dt} &= [\vec{\Psi}, H]_C = \frac{1}{2} [i(\vec{\sigma} \times \vec{D} \Psi_0 + \vec{D} \times \vec{\Psi}) - \vec{\sigma} \times (\vec{\sigma} \times \vec{D} \Psi_0 + \vec{D} \times \vec{\Psi})] \\ &= \vec{D} \Psi_0 + \frac{1}{2} [-\vec{\sigma} \times (\vec{D} \times \vec{\Psi}) + i\vec{D} \times \vec{\Psi}] \quad , \end{aligned} \tag{89}$$



we obtain the  $\vec{\Psi}$  equation of motion in the form given in Eq. (27). Similarly, from the bracket of  $\vec{P}$  with  $H$  we find the equation of motion for  $\vec{\Psi}^\dagger$ . Turning to brackets of the constraints with  $H$ , starting with  $P_{\Psi_0^\dagger}$ , we find

$$\frac{dP_{\Psi_0^\dagger}}{dt} = [P_{\Psi_0^\dagger}, H]_C = -\frac{1}{2}\chi \quad , \quad (90)$$

and so  $P_{\Psi_0^\dagger} = 0$  for all times implies that  $\chi = 0$ . For the total time derivative of  $\chi$ , we have

$$\frac{d\chi}{dt} = \frac{\partial\chi}{\partial t} + [\chi, H]_C = \vec{\sigma} \times g \frac{\partial \vec{A}}{\partial t} \cdot \vec{\Psi} + [\chi, H]_C = -ig\omega \quad , \quad (91)$$

and so  $\chi = 0$  for all times implies that  $\omega$  defined in Eq. (26) vanishes. Since  $\omega$  contains a term proportional to  $\Psi_0$ , to continue this process by calculating the time derivative of  $\omega$ , we must obtain  $d\Psi_0/dt$  from a bracket of  $\Psi_0$  with  $H$  (and similarly for  $d\Psi_0^\dagger/dt$ ). This requires adding to  $H$  a term

$$\Delta H = - \int d^3x \left[ P_{\Psi_0} \frac{d\Psi_0}{dt} + P_{\Psi_0^\dagger} \frac{d\Psi_0^\dagger}{dt} \right] \quad . \quad (92)$$

Requiring  $\Delta H$  to be self-adjoint then imposes the requirement

$$P_{\Psi_0}^\dagger = -P_{\Psi_0^\dagger} \quad , \quad (93)$$

which was noted following Eq. (72). We remark again that the chain of successive brackets with  $H$  starting from  $P_{\Psi_0^\dagger}$  and continuing to  $\chi, \omega, \dots$  leads only to constraints involving  $\vec{\Psi}$  and  $\Psi_0$  but never their adjoints. The doubling of the set of constraints, which turns the first class constraints into second class ones, comes from requiring that the adjoint of each fermionic constraint also be a constraint, not from taking successive brackets with  $H$ .

We are now ready to implement the Dirac bracket procedure. The basic idea is to change the canonical bracket  $[F, G]_C$  to a modified bracket  $[F, G]_D$ , which projects  $F$  and  $G$  onto the subspace obeying the constraints, so that the constraints are built into the brackets, or after quantization, into the canonical anticommutators. The constraints can then be “strongly” implemented in the Hamiltonian by setting terms proportional to the constraints to zero. In doing this we will choose the gauge  $\Psi_0 = \Psi_0^\dagger = 0$ , as we did in the path integral discussion. Since after integration by parts the second line of Eq. (88) takes the form

$$H = -\frac{1}{2} \int d^3x [-\Psi_0^\dagger \chi - \chi^\dagger \Psi_0 + \vec{\Psi}^\dagger \cdot \vec{D} \times \vec{\Psi}] \quad , \quad (94)$$

this eliminates problems associated with the fact that  $\Psi_0, \Psi_0^\dagger$  have as coefficients the constraints  $\chi^\dagger, \chi$  respectively. Setting constraint terms to zero in Eq. (94) we see that the constrained

Hamiltonian is just

$$\begin{aligned}
H &= -\frac{1}{2} \int d^3x \vec{\Psi}^\dagger \cdot \vec{D} \times \vec{\Psi} \\
&= -\frac{1}{2} \int d^3x (i\vec{P} - \vec{P} \times \vec{\sigma}) \cdot \vec{D} \times \vec{\Psi}
\end{aligned} \tag{95}$$

which coincides with the energy integral computed in Eq. (33) from the stress-energy tensor.

We proceed now to calculate the Dirac bracket for the case when  $F = F(\vec{\Psi})$  and  $G = G(\vec{\Psi}, \vec{\Psi}^\dagger)$ ; the case when  $F = F(\vec{\Psi}^\dagger)$  can then be obtained by taking the adjoint, and the case when  $F = F(\vec{\Psi}, \vec{\Psi}^\dagger)$  can be obtained by combining the extra bracket terms from both calculations. When  $F$  has no dependence on  $\vec{\Psi}^\dagger$ , it has vanishing brackets with the constraints  $\phi_a$  of Eq. (71) and nonvanishing brackets with the constraints  $\chi_a$  of Eq. (72). The Dirac bracket then has the form (see Eqs. (A20) and (A21) for why  $M^{-1}$  appears)

$$[F, G]_D = [F, G]_C - \sum_a \sum_b [F, \chi_a]_C M_{ab}^{-1} [\phi_b, G] \quad , \tag{96}$$

where  $M_{ab}(\vec{x}, \vec{y}) = [\phi_a(\vec{x}), \chi_b(\vec{y})]_C$  is the matrix defined in Eqs. (74) and (75). We recall that this matrix has the form

$$M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & \mathcal{U} & \mathcal{S} & \mathcal{T} \\ 0 & \mathcal{V} & \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{W} & \mathcal{C} & \mathcal{D} \end{pmatrix} \quad , \tag{97}$$

where in the  $SU(n)$  gauge field case, each entry in  $M$  is a  $2n \times 2n$  matrix. Using the block inversion method given in Eqs. (A18) and (A19), we find that  $M^{-1}$  is given by

$$M^{-1} = \begin{pmatrix} \Sigma & 1 & -(\mathcal{S}\mathcal{F} + \mathcal{T}\mathcal{H}) & -(\mathcal{S}\mathcal{G} + \mathcal{T}\mathcal{I}) \\ -1 & 0 & 0 & 0 \\ \mathcal{F}\mathcal{V} + \mathcal{G}\mathcal{W} & 0 & \mathcal{F} & \mathcal{G} \\ \mathcal{H}\mathcal{V} + \mathcal{I}\mathcal{W} & 0 & \mathcal{H} & \mathcal{I} \end{pmatrix} \quad , \tag{98}$$

where

$$\Sigma = \mathcal{U} - \mathcal{S}(\mathcal{F}\mathcal{V} + \mathcal{G}\mathcal{W}) - \mathcal{T}(\mathcal{H}\mathcal{V} + \mathcal{I}\mathcal{W}) \quad , \tag{99}$$

and where  $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}$  are the elements of the block inversion of the matrix  $N$  of Eq. (76),

$$\begin{pmatrix} \mathcal{F} & \mathcal{G} \\ \mathcal{H} & \mathcal{I} \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad . \tag{100}$$

Substituting these into Eq. (96) we find for the Dirac bracket a lengthy expression, which simplifies considerably after noting that  $[F(\vec{\Psi}), \chi_1]_C = [F(\vec{\Psi}), -P_{\Psi_0}]_C = 0$  and  $[\phi_1, G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C = [P_{\Psi_0^\dagger}, G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C = 0$ , leaving the relatively simple formula

$$\begin{aligned} [F(\vec{\Psi}), G(\vec{\Psi}, \vec{\Psi}^\dagger)]_D = & [F(\vec{\Psi}), G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C \\ & - [F(\vec{\Psi}), \chi_3]_C \left( \mathcal{F} [\phi_3, G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C + \mathcal{G} [\phi_4, G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C \right) \\ & - [F(\vec{\Psi}), \chi_4]_C \left( \mathcal{H} [\phi_3, G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C + \mathcal{I} [\phi_4, G(\vec{\Psi}, \vec{\Psi}^\dagger)]_C \right) . \end{aligned} \quad (101)$$

We note that just as in the path integral derivation, only the matrix  $N$  enters, in this case through its inverse, rather than the full matrix of constraint brackets  $M$ . The final step is to evaluate the inverse block matrix elements  $\mathcal{F}$ ,  $\mathcal{G}$ ,  $\mathcal{H}$ ,  $\mathcal{I}$  from the expressions for  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ ,  $\mathcal{D}$ , again by using the block inversion formulas of Eqs. (A18) and (A19). Let us define the Green's function  $\mathcal{D}^{-1}(\vec{x} - \vec{y})$  by

$$(i(\vec{L}_{\vec{x}})^2 + \vec{\sigma} \cdot \vec{L}_{\vec{x}} \times \vec{L}_{\vec{x}}) \mathcal{D}^{-1}(\vec{x} - \vec{y}) = \delta^3(\vec{x} - \vec{y}) \quad , \quad (102)$$

and a second Green's function  $\mathcal{Z}(\vec{x} - \vec{y})$  by

$$\begin{aligned} \mathcal{Z}(\vec{x} - \vec{y}) = & \mathcal{A} - \mathcal{B} \mathcal{D}^{-1} \mathcal{C} \\ = & -2ig\vec{\sigma} \cdot \vec{B} \delta^3(\vec{x} - \vec{y}) - 4\vec{D}_{\vec{x}} \cdot \vec{L}_{\vec{x}} \mathcal{D}^{-1}(\vec{x} - \vec{y}) \vec{L}_{\vec{y}} \cdot \overleftarrow{\vec{D}}_{\vec{y}} \quad . \end{aligned} \quad (103)$$

where we recall that for  $\Psi_0 = 0$  gauge  $\vec{L} = \vec{B} + \vec{\sigma} \times \vec{E}$ . Then the needed inverse block matrices are

$$\begin{aligned} \mathcal{F} = & \mathcal{Z}^{-1} \quad , \\ \mathcal{G} = & -\mathcal{Z}^{-1} \mathcal{B} \mathcal{D}^{-1} \quad , \\ \mathcal{H} = & -\mathcal{D}^{-1} \mathcal{C} \mathcal{Z}^{-1} \quad , \\ \mathcal{I} = & \mathcal{D}^{-1} + \mathcal{D}^{-1} \mathcal{C} \mathcal{Z}^{-1} \mathcal{B} \mathcal{D}^{-1} \quad . \end{aligned} \quad (104)$$

We wish now to apply the Dirac bracket formula to the cases (i)  $F(\vec{\Psi}) = \vec{\Psi}$  and  $G(\vec{\Psi}, \vec{\Psi}^\dagger) = \vec{\Psi}^\dagger$ , and (ii)  $F(\vec{\Psi}) = \vec{\Psi}$  and  $G(\vec{\Psi}, \vec{\Psi}^\dagger) = H$ , with  $H$  the constrained Hamiltonian of Eq. (95). The

following canonical brackets are needed for this:

$$\begin{aligned}
[\vec{\Psi}(\vec{x}), \chi_3(\vec{y})]_C &= 2\vec{D}_{\vec{x}}\delta^3(\vec{x} - \vec{y}) \quad , \\
[\vec{\Psi}(\vec{x}), \chi_4(\vec{y})]_C &= (i\vec{L}_{\vec{x}} - \vec{\sigma} \times \vec{L}_{\vec{x}})\delta^3(\vec{x} - \vec{y}) \quad , \\
[\phi_3(\vec{x}), \vec{\Psi}^\dagger(\vec{y})]_C &= 2\vec{D}_{\vec{x}}\delta^3(\vec{x} - \vec{y}) = -2\delta^3(\vec{x} - \vec{y})\overleftarrow{D}_{\vec{y}} \quad , \\
[\phi_4(\vec{x}), \vec{\Psi}^\dagger(\vec{y})]_C &= -(i\vec{L}_{\vec{x}} - \vec{L}_{\vec{x}} \times \vec{\sigma})\delta^3(\vec{x} - \vec{y}) = \delta^3(\vec{x} - \vec{y})(i\overleftarrow{L}_{\vec{y}} - \overleftarrow{L}_{\vec{y}} \times \vec{\sigma}) \quad , \\
[\phi_3(\vec{x}), H]_C &= ig\vec{B}(\vec{x}) \cdot \vec{\Psi}(\vec{x}) \quad , \\
[\phi_4(\vec{x}), H]_C &= \frac{1}{2}(i\vec{L}_{\vec{x}} - \vec{L}_{\vec{x}} \times \vec{\sigma}) \times \vec{D}_{\vec{x}} \cdot \vec{\Psi}(\vec{x}) \quad .
\end{aligned} \tag{105}$$

Additionally, for case (i) we need the canonical bracket

$$\begin{aligned}
[\Psi_i(\vec{x}), \Psi_j^\dagger(\vec{y})]_C &= [\Psi_i(\vec{x}), iP_j(\vec{y}) - \epsilon_{jkl}P_k(\vec{y})\sigma_l]_C \\
&= -i(\delta_{ij} + i\epsilon_{jil}\sigma_l)\delta^3(\vec{x} - \vec{y}) = -i\sigma_j\sigma_i\delta^3(\vec{x} - \vec{y}) = -2i\left(\delta_{ij} - \frac{1}{2}\sigma_i\sigma_j\right)\delta^3(\vec{x} - \vec{y}) \quad ,
\end{aligned} \tag{106}$$

and for case (ii) we need the canonical bracket

$$[\Psi_i(\vec{x}), H]_C = \frac{1}{2}\left(i\vec{D}_{\vec{x}} \times \vec{\Psi}(\vec{x}) - \vec{\sigma} \times (\vec{D}_{\vec{x}} \times \vec{\Psi}(\vec{x}))\right)_i \quad . \tag{107}$$

Up to this point, we have not specialized  $\vec{L}$  so as to make it easy to ascertain what the formulas become when gauge fixing is omitted (as in [3] and [4]). When  $\vec{L} = 0$ , the matrix  $N$  degenerates to its upper left element  $\mathcal{A}$ , which is a local function of  $\vec{x}$  and so is algebraically invertible. For the Dirac bracket of  $\vec{\Psi}_i(\vec{x})$  with  $\vec{\Psi}_j^\dagger(\vec{y})$  we then find

$$\begin{aligned}
[\Psi_i(\vec{x}), \Psi_j^\dagger(\vec{y})]_D &= [\Psi_i(\vec{x}), \Psi_j^\dagger(\vec{y})]_C - \int d^3w d^3z [\Psi_i(\vec{x}), \chi_3(\vec{w})]_C \mathcal{Z}^{-1}(\vec{w} - \vec{z}) [\phi_3(\vec{z}), \Psi_j^\dagger(\vec{y})]_C \\
&= -2i\left[\left(\delta_{ij} - \frac{1}{2}\sigma_i\sigma_j\right)\delta^3(\vec{x} - \vec{y}) - D_{\vec{x}i}\frac{\delta^3(\vec{x} - \vec{y})}{g\vec{\sigma} \cdot \vec{B}(\vec{x})}\overleftarrow{D}_{\vec{y}j}\right] \\
&= -2i\langle\vec{x}|\left[\left(\delta_{ij} - \frac{1}{2}\sigma_i\sigma_j\right)1 + \Pi_i\frac{1}{g\vec{\sigma} \cdot \vec{B}}\Pi_j\right]|\vec{y}\rangle \quad ,
\end{aligned} \tag{108}$$

where in the final line we have written  $iD_{\vec{x}i} = \Pi_i$  to relate to the abstract operator notation of Velo and Zwanziger [4]. Multiplying the final line by  $i$  to convert the Dirac bracket to an anticommutator, and by a factor 1/2 reflecting our different field normalization, Eq. (108) becomes the expression for

the anticommutator given in the zero mass limit of Eq. (4.12) of [4]. Using identities in Appendix A, one can verify (as in Appendix C of [4]) that

$$(\vec{\sigma} \times \vec{D}_{\vec{x}})_i \left[ \left( \delta_{ij} - \frac{1}{2} \sigma_i \sigma_j \right) \delta^3(\vec{x} - \vec{y}) - D_{\vec{x}i} \frac{\delta^3(\vec{x} - \vec{y})}{g\vec{\sigma} \cdot \vec{B}(\vec{x})} \overleftarrow{D}_{\vec{y}j} \right] = 0 \quad , \quad (109)$$

that is, the constraint  $\chi$  is explicitly projected to zero.

Now setting  $\vec{L} = \vec{B} + \vec{\sigma} \times \vec{E}$ , so that the  $\omega$  constraint implies  $\Psi_0 = \Psi_0^\dagger = 0$ , we find for the Dirac bracket of  $\Psi_i(\vec{x})$  with the constrained Hamiltonian

$$\begin{aligned} \frac{d\vec{\Psi}(\vec{x})}{dt} = & [\vec{\Psi}(\vec{x}), H]_D = \frac{1}{2} [i\vec{D}_{\vec{x}} \times \vec{\Psi}(\vec{x}) - \vec{\sigma} \times (\vec{D}_{\vec{x}} \times \vec{\Psi}(\vec{x}))] \\ & - \int d^3y \left\{ 2\vec{D}_{\vec{x}} \left[ \mathcal{F}(\vec{x} - \vec{y}) ig\vec{B}(\vec{y}) \cdot \vec{\Psi}(\vec{y}) \right. \right. \\ & + \mathcal{G}(\vec{x} - \vec{y}) \frac{1}{2} (i\vec{L}_{\vec{y}} - \vec{L}_{\vec{y}} \times \vec{\sigma}) \times \vec{D}_{\vec{y}} \cdot \vec{\Psi}(\vec{y}) \Big] \\ & + (i\vec{L}_{\vec{x}} - \vec{\sigma} \times \vec{L}_{\vec{x}}) \left[ \mathcal{H}(\vec{x} - \vec{y}) ig\vec{B}(\vec{y}) \cdot \vec{\Psi}(\vec{y}) \right. \\ & \left. \left. + \mathcal{I}(\vec{x} - \vec{y}) \frac{1}{2} (i\vec{L}_{\vec{y}} - \vec{L}_{\vec{y}} \times \vec{\sigma}) \times \vec{D}_{\vec{y}} \cdot \vec{\Psi}(\vec{y}) \right] \right\} \quad . \end{aligned} \quad (110)$$

The first line of this equation is the unconstrained equation of motion in the form of Eq. (27) (when  $A_0 = \Psi_0 = 0$ ), while the remaining terms guarantee that

$$\begin{aligned} \frac{d\phi_3}{dt} = \frac{d\chi}{dt} = \frac{d(\vec{\sigma} \times \vec{D} \cdot \vec{\Psi})}{dt} = \vec{\sigma} \times \vec{D} \cdot \frac{d\vec{\Psi}}{dt} = 0 \quad , \\ \frac{d\phi_4}{dt} = d \frac{\vec{L} \cdot \vec{\Psi}}{dt} = \vec{L} \cdot \frac{d\vec{\Psi}}{dt} = 0 \quad , \end{aligned} \quad (111)$$

where we have used the fact that we are assuming that  $\vec{D}$  and  $\vec{L}$  are time independent. This restriction can be avoided by treating the gauge fields as dynamical variables, taking into account their own constraint structure, and noting that the radiation gauge fixing constraint  $\vec{\nabla} \cdot \vec{P}_{\vec{A}} = 0$ , with  $\vec{P}_{\vec{A}}$  the canonical momentum conjugate to  $\vec{A}$ , has nonvanishing fermionic brackets with all Rarita-Schwinger constraints involving  $\vec{D} = \vec{\nabla} + g\vec{A}$ . This requires an extension of the Dirac bracket construction to take the new, Grassmann-odd, brackets into account, and the extended Dirac bracket structure will then obey Eq. (111) without requiring the assumption of a time independent  $\vec{A}$  and  $\vec{L}$ .

In the next section, where we show positivity of the Dirac anticommutator, we will use covariant radiation gauge instead of  $\Psi_0 = 0$  gauge. Setting  $\vec{L} = \vec{D}$  and putting everything together, we find

for the Dirac bracket of  $\Psi_i(\vec{x})$  with  $\Psi_j^\dagger(\vec{y})$ ,

$$\begin{aligned}
[\Psi_i(\vec{x}), \Psi_j^\dagger(\vec{y})]_D = & -2i\left(\delta_{ij} - \frac{1}{2}\sigma_i\sigma_j\right)\delta^3(\vec{x} - \vec{y}) \\
& + 4\vec{D}_{\vec{x}i}\mathcal{F}(\vec{x} - \vec{y})\overleftarrow{D}_{\vec{y}j} - 2D_{\vec{x}i}\mathcal{G}(\vec{x} - \vec{y})(i\overleftarrow{D}_{\vec{y}} - \overleftarrow{D}_{\vec{y}} \times \vec{\sigma})_j \\
& + 2(i\vec{D}_{\vec{x}} - \vec{\sigma} \times \vec{D}_{\vec{x}})_i\mathcal{H}(\vec{x} - \vec{y})\overleftarrow{D}_{\vec{y}j} - (i\vec{D}_{\vec{x}} - \vec{\sigma} \times \vec{D}_{\vec{x}})_i\mathcal{I}(\vec{x} - \vec{y})(i\overleftarrow{D}_{\vec{y}} - \overleftarrow{D}_{\vec{y}} \times \vec{\sigma})_j \quad ,
\end{aligned} \tag{112}$$

which gives the generalization of Eq. (108) to the case when a covariant gauge fixing constraint is imposed.

## IX. QUANTIZATION OF THE ANTICOMMUTATOR DERIVED FROM THE DIRAC BRACKET AND POSITIVITY IN COVARIANT RADIATION GAUGE

Given the Dirac bracket, the next step is to quantize, by multiplying all Dirac brackets by  $i$  and then reinterpreting them as anticommutators or commutators of operators. In the case considered here, this can be done in a constructive way, as follows. First let us replace the set of  $2n$  component column vector constraints  $\phi_a$  and  $2n$  component row vector constraints  $\chi_a$  by the set of  $4n$  scalars given by their individual matrix elements. Moreover, since the  $\chi_a$  are the adjoints of the  $\phi_a$ , we can take linear combinations to make all of these scalars self-adjoint. Labeling the set of self-adjoint scalar constraints by  $\Phi_a$ , the Dirac bracket construction for the bracket of  $F$  with  $G$  reads

$$\begin{aligned}
[F, G]_D = & [F, G]_C - \sum_a \sum_b [F, \Phi_a]_C T_{ab}^{-1} [\Phi_b, G]_C \quad , \\
T_{ab} = & [\Phi_a, \Phi_b]_C \quad ,
\end{aligned} \tag{113}$$

with the matrix  $T$  real.

We now observe that since the  $\Phi_a$  are all *linear* in the scalar components of  $\vec{\Psi}$  and  $\vec{\Psi}^\dagger$ , if we make the replacement  $i[\ , \ ]_C \rightarrow \{ \ , \ \}_C$ , with  $\{ \ , \ \}$  the anticommutator, and replace all Grassmann variables  $\vec{\Psi}$  and  $\vec{\Psi}^\dagger$  with operator variables having the standard canonical anticommutators, then since there is no other operator structure the same real matrix  $T_{ab}$  will be obtained. Moreover, if  $F$  and  $G$  are both linear in the scalar components of  $\vec{\Psi}$  and  $\vec{\Psi}^\dagger$ , the Grassmann bracket  $i[F, G]_C$  formed from scalar components of  $F$  and  $G$  will agree with the canonical anticommutator  $i\{F, G\}_C$  formed from the corresponding operator scalar components, and will be a  $c$ -number. Thus, for

linear  $F$  and  $G$  we can define a “Dirac anticommutator”  $\{F, G\}_D$  by

$$\begin{aligned}\{F, G\}_D &= \{F, G\}_C - \sum_a \sum_b \{F, \Phi_a\}_C T_{ab}^{-1} \{\Phi_b, G\}_C \quad , \\ T_{ab} &= \{\Phi_a, \Phi_b\}_C \quad .\end{aligned}\tag{114}$$

When one or both of  $F$  and  $G$  is bilinear, the Grassmann bracket  $i[F, G]_C$  formed from the scalar components of  $F$  and  $G$  will agree with the canonical commutator formed from the corresponding operator scalar components, and we can define a “Dirac commutator” by a formula analogous to Eq. (114) in which each anticommutator with at least one bilinear argument is replaced by a commutator. In this way we get a mapping of classical brackets into quantum anticommutators and commutators, that inherits the algebraic properties of the Dirac bracket, including the chain rule, with the Jacobi identities for odd and even Grassmann variables mapping to the corresponding anticommutator and commutator Jacobi identities.

To complete this correspondence, we must show that the Dirac anticommutator of  $\Psi_i^{\alpha u}$  and  $\Psi_j^{\dagger \beta v}$  (with  $\alpha = 1, 2$ ,  $\beta = 1, 2$  the spin indices,  $u = 1, \dots, n$ ,  $v = 1, \dots, n$  the internal symmetry indices, and  $i = 1, 2, 3$ ,  $j = 1, 2, 3$  the spatial vector indices) has the expected positivity properties of an operator anticommutator, by showing that for an arbitrary set of complex functions  $A_i^{\alpha u}(\vec{x})$ , we have

$$\int d^3x d^3y A_i^{\alpha u}(\vec{x}) A_j^{* \beta v}(\vec{y}) \{\Psi_i^{\alpha u}(\vec{x}), \Psi_j^{\dagger \beta v}(\vec{y})\}_D \geq 0 \quad .\tag{115}$$

We demonstrate this in several steps, in covariant radiation gauge. First we examine the conditions for positivity of the canonical anticommutator and Poisson bracket,

$$\int d^3x d^3y A_i^{\alpha u}(\vec{x}) A_j^{* \beta v}(\vec{y}) \{\Psi_i^{\alpha u}(\vec{x}), \Psi_j^{\dagger \beta v}(\vec{y})\}_C = \int d^3x d^3y A_i^{\alpha u}(\vec{x}) A_j^{* \beta v}(\vec{y}) i[\Psi_i^{\alpha u}(\vec{x}), \Psi_j^{\dagger \beta v}(\vec{y})]_C \quad .\tag{116}$$

From  $\Psi_j^{\dagger \beta v} = iP_j^{\beta v} - \epsilon_{jkl} P_k^{\delta v} \sigma_l^{\delta \beta}$ , we find that

$$\begin{aligned}[\Psi_i^{\alpha u}(\vec{x}), \Psi_j^{\dagger \beta v}(\vec{y})]_C &= -i(\delta_{ij} \delta^{\alpha \beta} + i\epsilon_{jik} \sigma_k^{\alpha \beta}) \delta^{uv} \delta^3(\vec{x} - \vec{y}) \\ &= -i(\sigma_j \sigma_i)^{\alpha \beta} \delta^{uv} \delta^3(\vec{x} - \vec{y}) = -2i(\delta_{ij} - \frac{1}{2} \sigma_i \sigma_j)^{\alpha \beta} \delta^{uv} \delta^3(\vec{x} - \vec{y}) \quad .\end{aligned}\tag{117}$$

Multiplying by  $i/2$ , and writing  $A_i^{\alpha u} = R_i^{\alpha u} + iI_i^{\alpha u}$ ,  $i = 1, 2, 3$ ,  $\alpha = 1, 2$ ,  $u = 1, \dots, n$ , with  $R$  and  $I$  real, the right hand side of Eq. (116) evaluates to (we suppress the internal symmetry index  $u$

from here on, so  $(R_i^\alpha)^2$  means  $\sum_{u=1}^n (R_i^{\alpha u})^2$ , etc.)

$$\sum_{i=1}^3 \sum_{\alpha=1}^2 ((R_i^\alpha)^2 + (I_i^\alpha)^2) - \frac{1}{2} ((R_2^1 - I_1^1 + I_3^2)^2 + (R_1^1 + I_2^1 - R_3^2)^2 + (R_2^2 + I_1^2 + I_3^1)^2 + (R_1^2 - I_2^2 + R_3^1)^2) \quad . \quad (118)$$

If all three components  $A_i^\alpha$ ,  $i = 1, \dots, 3$  are present, the expression in Eq. (118) is *not* positive semidefinite. But when only two of the three components are present, as a result of application of a constraint, then each of the four squared terms on the right hand side of Eq. (118) contains only two terms, and so the expression in Eq. (118) is positive semidefinite by virtue of the inequality

$$X^2 + Y^2 - \frac{1}{2}(X \pm Y)^2 = \frac{1}{2}(X \mp Y)^2 \geq 0 \quad . \quad (119)$$

Another way of seeing this, noted by both Velo and Zwanziger [4] and Allcock and Hall [10], is that because  $\sum_{i=1}^3 \sigma_i \sigma_i = 3$ , the expression  $W_{ij} = \delta_{ij} - \frac{1}{2} \sigma_i \sigma_j$  is not a projector. But when one component of  $\vec{\sigma}$ , say  $\sigma_3$ , is replaced by 0, so that one has  $\sum_{i=1}^3 \sigma_i \sigma_i = \sum_{i=1}^2 \sigma_i \sigma_i = 2$ , then

$$\sum_l W_{il} W_{lj} = \delta_{ij} - 2 \frac{1}{2} \sigma_i \sigma_j + \frac{1}{4} \sigma_i \sum_{l=1}^2 \sigma_l \sigma_l = \delta_{ij} - \frac{1}{2} \sigma_i \sigma_j = W_{ij} \quad , \quad (120)$$

and  $W_{ij}$  is a projector and hence is positive semidefinite. So we anticipate that proving positivity will require projection of Eq. (117) into a subspace obeying at least one constraint on  $\vec{\Psi}$ .

The next step is to use the property that the Dirac bracket of linear quantities  $F$  and  $G$  reduces to the canonical bracket of their projections into the subspace obeying the constraints, when (as is the case here) all constraints are second class, that is they all appear in the Dirac bracket [18]. Referring to Eq. (113), let us define

$$\begin{aligned} \tilde{F} &= F - \sum_a \sum_b [F, \Phi_a]_C T_{ab}^{-1} \Phi_b \quad , \\ \tilde{G} &= G - \sum_a \sum_b [G, \Phi_a]_C T_{ab}^{-1} \Phi_b \quad , \end{aligned} \quad (121)$$

so that

$$\begin{aligned} [\tilde{F}, \Phi_c]_C &= [F, \Phi_c]_C - \sum_a \sum_b [F, \Phi_a]_C T_{ab}^{-1} [\Phi_b, \Phi_c]_C \\ &= [F, \Phi_c]_C - \sum_a \sum_b [F, \Phi_a]_C T_{ab}^{-1} T_{bc} \\ &= [F, \Phi_c]_C - \sum_a [F, \Phi_a]_C \delta_{ac} = 0 \quad , \end{aligned} \quad (122)$$



and similarly for  $\tilde{G}$ . As a result of this relation, which holds when the canonical brackets are simply numbers (as in the case here where  $\Phi_c$  and  $F, G$  are linear), together with symmetry of the canonical bracket  $[\tilde{G}, \Phi_c]_C = [\Phi_c, \tilde{G}]_C$ , we see that

$$[F, G]_D = [\tilde{F}, \tilde{G}]_C \quad . \quad (123)$$

These properties of Eqs. (121)–(123) carry over when we replace Grassmann numbers with operators, and classical brackets with anticommutators, since in the linear case all anticommutators of linear quantities are c-numbers that commute with the operators, and since the anticommutator is symmetric. Thus we have

$$\{\Psi_i^\alpha(\vec{x}), \Psi_j^{\dagger\beta}(\vec{y})\}_D = \{\tilde{\Psi}_i^\alpha(\vec{x}), \tilde{\Psi}_j^{\dagger\beta}(\vec{y})\}_C \quad . \quad (124)$$

To further study the properties of  $\tilde{\Psi}_i(\vec{x})$  and  $\tilde{\Psi}_j^\dagger(\vec{y})$  (with spinor indices suppressed), let us now return to our original labeling of the constraints by  $\phi_a$  and  $\chi_a$  as in Eq. (101), so that we have in the Dirac bracket formalism

$$\tilde{\Psi}_i(\vec{x}) = \Psi_i(\vec{x}) - \sum_a \sum_b [\Psi_i(\vec{x}), \chi_a]_C M_{ab}^{-1} \phi_b \quad , \quad (125)$$

and a similar equation (with the roles of  $\phi_a$  and  $\chi_a$  interchanged) for  $\tilde{\Psi}_j^\dagger(\vec{y})$ , with  $a, b$  summed from 3 to 4. We now note two important properties of this equation. The first is that it is invariant under replacement of the constraints  $\chi_a$  by any linear combination  $\chi'_a = \chi_b K_{ba}$ , with the matrix  $K$  nonsingular, since the factors  $K$  and  $K^{-1}$  cancel between  $\chi'_a$  and  $M_{ab}'^{-1}$ . (More generally, the Dirac bracket is invariant under replacement of the constraints by any nonsingular linear combination of the constraints, reflecting the fact that the Dirac bracket is a projector onto the subspace obeying the constraints, and this subspace is invariant under replacement of the constraints by any nonsingular linear combination of the constraints.) The second is that if we act on  $\tilde{\Psi}_i(\vec{x})$  with either  $D_{\vec{x}i}$  or  $(\vec{\sigma} \times D_{\vec{x}})_i$ , we get zero. For example, recalling that in covariant radiation gauge  $D_{\vec{x}i} \Psi_i(\vec{x}) = \phi_4(\vec{x})$ , we have (with spatial variable labels  $\vec{x}$  suppressed)

$$D_i \tilde{\Psi}_i = \phi_4 - \sum_a \sum_b [\phi_4, \chi_a]_C M_{ab}^{-1} \phi_b = \phi_4 - \sum_a \sum_b M_{4a} M_{ab}^{-1} \phi_b = \phi_4 - \sum_b \delta_{4b} \phi_b = 0 \quad , \quad (126)$$

and similarly for  $(\vec{\sigma} \times D_{\vec{x}})_i$ , with  $\phi_4$  replaced by  $\phi_3$ .

Let us now write  $\tilde{\Psi}_i(\vec{x})$  as a projector  $R_{ij}(\vec{x}, \vec{y})$  acting on  $\Psi_j(\vec{y})$ , giving after an integration by

parts on  $\vec{y}$ ,

$$\begin{aligned}\tilde{\Psi}_i(\vec{x}) &= \int d^3y R_{ij}(\vec{x}, \vec{y}) \Psi_j(\vec{y}) \quad , \\ R_{ij}(\vec{x}, \vec{y}) &= \delta_{ij} \delta^3(\vec{x} - \vec{y}) + \sum_a \sum_b \int d^3z [\Psi_i(\vec{x}), \chi_a(\vec{z})]_C M_{ab}^{-1}(\vec{z}, \vec{y}) \overleftarrow{\eta}_{bj}(\vec{y}) \quad ,\end{aligned}\tag{127}$$

with

$$\overleftarrow{\eta}_{3j}(\vec{y}) = (\vec{\sigma} \times \overleftarrow{D}_{\vec{y}})_j \quad , \quad \overleftarrow{\eta}_{4j}(\vec{y}) = \overleftarrow{D}_{\vec{y}j} \quad .\tag{128}$$

By virtue of Eq. (126) and its analog for  $\vec{\sigma} \times \vec{D}$ , we have

$$\begin{aligned}D_{\vec{x}i} R_{ij}(\vec{x}, \vec{y}) &= 0 \quad , \\ (\vec{\sigma} \times \vec{D}_{\vec{x}})_i R_{ij}(\vec{x}, \vec{y}) &= 0 \quad ,\end{aligned}\tag{129}$$

and by the reasoning of Eqs. (40)–(42) we also have (assuming  $\vec{\sigma} \cdot \vec{D}$  is invertible)

$$\sigma_i R_{ij}(\vec{x}, \vec{y}) = 0 \quad .\tag{130}$$

Next let us focus on the bracket  $[\Psi_i(\vec{x}), \chi_a(\vec{z})]_C$  appearing as the first factor inside the sum. Setting  $\vec{L} = \vec{D}$  in Eq. (105) we have

$$\begin{aligned}[\vec{\Psi}(\vec{x}), \chi_3(\vec{z})]_C &= 2\vec{D}_{\vec{x}} \delta^3(\vec{x} - \vec{z}) \quad , \\ [\vec{\Psi}(\vec{x}), \chi_4(\vec{z})]_C &= (i\vec{D}_{\vec{x}} - \vec{\sigma} \times \vec{D}_{\vec{x}}) \delta^3(\vec{x} - \vec{z}) \quad .\end{aligned}\tag{131}$$

Using the invariance of  $\tilde{\Psi}_i$ , or equivalently of  $R_{ij}$ , under replacement of  $\chi_3, \chi_4$  by any nondegenerate linear combination of  $\chi_3, \chi_4$ , let us choose the new combinations so that

$$\begin{aligned}[\vec{\Psi}(\vec{x}), \chi_3(\vec{z})]_C &= (\vec{\sigma} \times \vec{D}_{\vec{x}}) \delta^3(\vec{x} - \vec{z}) = \vec{\eta}_3(\vec{x}) \delta^3(\vec{x} - \vec{z}) \quad , \\ [\vec{\Psi}(\vec{x}), \chi_4(\vec{z})]_C &= \vec{D}_{\vec{x}} \delta^3(\vec{x} - \vec{y}) = \vec{\eta}_4(\vec{x}) \delta^3(\vec{x} - \vec{z}) \quad .\end{aligned}\tag{132}$$

Substituting this into Eq. (127), we get the symmetric expression

$$R_{ij}(\vec{x}, \vec{y}) = \delta_{ij} \delta^3(\vec{x} - \vec{y}) + \sum_a \sum_b \int d^3z \vec{\eta}_{ai}(\vec{x}) M_{ab}^{-1}(\vec{x}, \vec{y}) \overleftarrow{\eta}_{bj}(\vec{y}) \quad .\tag{133}$$

By virtue of this symmetry, the projector  $R_{ij}$  is annihilated by the constraints  $\overleftarrow{D}_{\vec{y}j}$  and  $(\vec{\sigma} \times \overleftarrow{D}_{\vec{y}})_j$  acting from the right, which in turn implies that in addition to Eq. (130) we also have

$$R_{ij}(\vec{x}, \vec{y})\sigma_j = 0 \quad . \quad (134)$$

An explicit construction of  $R_{ij}(\vec{x}, \vec{y})$  and verification of Eqs. (130) and (134) is given in Appendix C.

Returning now to Eqs. (115) and (124), writing  $\tilde{\Psi}_i^\alpha$  and  $\tilde{\Psi}_j^{\dagger\beta}$  in terms of projectors acting on  $\Psi_i^\alpha$  and  $\Psi_j^{\dagger\beta}$ , we have (using  $\sigma_m^{\epsilon\delta} = \sigma_m^{*\delta\epsilon}$ , and continuing to suppress internal symmetry indices  $u, v$ , which are contracted in the same pattern as the spatial vector and spin indices)

$$\begin{aligned} & \int d^3x \int d^3y A_i^\alpha(\vec{x}) A_j^{*\beta}(\vec{y}) \{ \Psi_i^\alpha(\vec{x}), \Psi_j^{\dagger\beta}(\vec{y}) \}_D \\ &= \int d^3x \int d^3y A_i^\alpha(\vec{x}) A_j^{*\beta}(\vec{y}) \{ \tilde{\Psi}_i^\alpha(\vec{x}), \tilde{\Psi}_j^{\dagger\beta}(\vec{y}) \}_C \\ &= \int d^3x \int d^3y A_i^\alpha(\vec{x}) A_j^{*\beta}(\vec{y}) \int d^3z \int d^3w R_{il}^{\alpha\gamma}(\vec{x}, \vec{z}) \{ \Psi_l^\gamma(\vec{z}), \Psi_m^{\dagger\delta}(\vec{w}) \}_C R_{jm}^{*\beta\delta}(\vec{y}, \vec{w}) \\ &= \int d^3x \int d^3y A_i^\alpha(\vec{x}) A_j^{*\beta}(\vec{y}) \int d^3z \int d^3w R_{il}^{\alpha\gamma}(\vec{x}, \vec{z}) 2 \left( \delta_{lm} \delta^{\gamma\delta} - \frac{1}{2} \sigma_l^{\gamma\epsilon} \sigma_m^{*\delta\epsilon} \right) \delta^3(\vec{z} - \vec{w}) R_{jm}^{*\beta\delta}(\vec{y}, \vec{w}) \\ &= 2 \int d^3z \left[ \int d^3x A_i^\alpha(\vec{x}) R_{il}^{\alpha\gamma}(\vec{x}, \vec{z}) \right] \left[ \int d^3y A_j^\beta(\vec{y}) R_{jl}^{\beta\gamma}(\vec{y}, \vec{z}) \right]^* , \end{aligned} \quad (135)$$

which is positive semidefinite.

We conclude that the anticommutator of  $\vec{\Psi}$  with  $\vec{\Psi}^\dagger$  is positive semidefinite in covariant radiation gauge. The symmetry of the  $\phi_{3,4}$  and  $\chi_{3,4}$  constraints in this gauge is essential to reaching this conclusion; if gauge fixing were omitted, or if another gauge were chosen, this symmetry would not be present and we could not deduce positivity in a similar fashion.

## X. LORENTZ COVARIANCE OF COVARIANT RADIATION AND $\Psi_0 = 0$ GAUGE AND LORENTZ INVARIANCE OF THE DIRAC BRACKET

We study next the behavior of covariant radiation gauge and the Dirac bracket under Lorentz boosts. The Rarita-Schwinger field  $\psi_\mu^\alpha$  and its left-handed chiral projection  $\Psi_\mu^\alpha$  both have a four-vector index  $\mu$  and a spinor index  $\alpha$ . Under an infinitesimal Lorentz transformation, the transformations acting on these two types of indices are additive, and so can be considered separately. The spinor indices are transformed as in the usual spin  $\frac{1}{2}$  Dirac equation by a matrix constructed from the Dirac gamma matrices, which commutes with  $D_\mu$ . Hence the spinor index transformation leaves the covariant radiation gauge condition  $\vec{D} \cdot \vec{\Psi}$  invariant.

This leaves the transformation on the vector index to be considered, and this is a direct analog of the Lorentz transformation of radiation gauge in quantum electrodynamics [19]. Since the radiation gauge condition is invariant under spatial rotations, we only have to consider a Lorentz boost,

$$\begin{aligned}\vec{x} \rightarrow \vec{x}' &= \vec{x} + \vec{v}t \quad , \\ x^0 = t \rightarrow t' &= t + \vec{v} \cdot \vec{x} \quad .\end{aligned}\tag{136}$$

Under this boost, the field  $\vec{\Psi}$  transforms as

$$\vec{\Psi} \rightarrow \vec{\Psi}' = \vec{\Psi} + \vec{v}\Psi^0 \quad .\tag{137}$$

For an observer in the boosted frame, covariant radiation gauge would be  $\vec{D}_{\vec{x}'} \cdot \vec{\Psi}' = 0$ , with  $\vec{D}_{\vec{x}'} = \vec{\nabla}_{\vec{x}'} + g\vec{A}'$ , where  $\vec{A}' = \vec{A} + O(\vec{v})$ . Applying this to  $\vec{\Psi}'(\vec{x}', t')$  and using the covariant radiation gauge condition in the initial frame, we get

$$\vec{D}_{\vec{x}'} \cdot \vec{\Psi}' = v_j \Sigma_j(\vec{x}, t) \quad ,\tag{138}$$

with  $\Sigma_j(\vec{x}, t)$  a local polynomial in  $\vec{\Psi}$ ,  $\Psi_0$  and the gauge fields, where we have dropped primes on the right hand side since there is an explicit factor of  $\vec{v}$ . So in the boosted frame  $\vec{\Psi}'$  does not obey the covariant radiation gauge condition, but this can be restored by making a gauge transformation

$$\vec{\Psi}' \rightarrow \vec{\Psi}' - \vec{D}(\vec{D}^2)^{-1} v_j \Sigma_j(\vec{x}, t) \quad .\tag{139}$$

Hence the covariant radiation gauge condition is Lorentz boost covariant, although not Lorentz boost invariant. Analogous arguments apply to  $\Psi_0 = 0$  gauge. Under the Lorentz boost of Eq. (136),  $\Psi_0 \rightarrow \Psi_0 - \vec{v} \cdot \vec{\Psi}$ , so  $\Psi_0 = 0$  gauge is not preserved, but it can be restored by a compensating infinitesimal gauge transformation.

Referring now to Eq. (C10), we note that the covariant radiation gauge Dirac bracket and the anticommutation relations are invariant under infinitesimal Rarita-Schwinger gauge transformations, such as that of Eq. (139), up to a remainder that is quadratic in the gauge parameter. Hence the covariant radiation gauge Dirac bracket and the anticommutation relations following from it are Lorentz invariant, since a finite Lorentz transformation can be built up from a series of infinitesimal ones.

## XI. COMPATIBILITY OF LORENTZ COVARIANCE AND MODE COUNTING IN ON-SHELL RARITA-SCHWINGER PARTICLE-PHOTON SCATTERING

We address finally the question [20] of whether leading order on-shell scattering of Rarita-Schwinger fields from an external electromagnetic field has the requisite relativistic covariance, while preserving the correct counting of massless spin  $\frac{3}{2}$  propagation modes. The operator effective action for this scattering process can be read off from the interaction term in Eq. (1),

$$\begin{aligned} S_{\text{eff}}(\psi_\mu, A_\nu) &= \int d^4x \mathcal{L}_{\text{eff}}(\psi_\mu, A_\nu) \quad , \\ \mathcal{L}_{\text{eff}}(\psi_\mu, A_\nu) &= \frac{1}{2} g \bar{\psi}_\mu(x) i \epsilon^{\mu\eta\nu\rho} \gamma_5 \gamma_\eta A_\nu(x) \psi_\rho(x) \quad , \end{aligned} \tag{140}$$

where we have suppressed spinor indices as in the text from Eq. (3) onwards. Since we are taking the external field  $A_\nu$  as Abelian, the covariant derivatives in the equations of motion and constraints are given by

$$D_\nu = \partial_\nu + g A_\nu \quad , \quad \overleftarrow{D}_\nu = \overleftarrow{\partial}_\nu - g A_\nu \quad . \tag{141}$$

To keep the analysis simple, we shall assume that  $A_\nu(x)$  is of short range, and vanishes for  $|\vec{x}| > R$  for some radius  $R$ . This effective action, the equations of motion of Eqs. (3) and (4), and the primary and secondary constraints following from them, given in Eqs. (5) and (6), are all relativistically covariant. Hence to do a covariant calculation, it is only necessary to choose a relativistically covariant gauge fixing, which for convenience we take as the gauge covariant Lorentz gauge condition

$$D^\mu \psi_\mu = 0 \quad , \tag{142}$$

which is attainable from a generic gauge by the gauge transformation of Eq. (9), provided that  $D^\mu D_\mu$  is invertible.

Taking the matrix element of Eq. (140) between an incoming Rarita-Schwinger state of four-momentum  $p$ , and an outgoing Rarita-Schwinger state of four momentum  $p'$ , we get the corresponding scattering amplitude

$$\mathcal{A} = \frac{1}{2} i g \int d^4x \bar{\psi}_\mu(p', x) \epsilon^{\mu\eta\nu\rho} \gamma_5 \gamma_\eta A_\nu(x) \psi_\rho(p, x) \quad , \tag{143}$$

where  $\psi_\rho$  and  $\bar{\psi}_\mu$  are now wave functions, rather than operators, that obey the Rarita-Schwinger equations of motion in the presence of the external field  $A_\nu$ .

We now introduce source currents for the gauge potential  $A_\nu$  and the Rarita-Schwinger wave functions  $\psi_\rho$  and  $\bar{\psi}_\mu$ , and study their conservation properties. The source current to which the gauge potential  $A_\nu$  couples is defined by writing the scattering amplitude as

$$\begin{aligned}\mathcal{A} &= \frac{1}{2}ig \int d^4x A_\nu(x) J^\nu(x) \quad , \\ J^\nu(x) &= \bar{\psi}_\mu(p', x) \epsilon^{\mu\eta\nu\rho} \gamma_5 \gamma_\eta \psi_\rho(p, x) \quad .\end{aligned}\tag{144}$$

The source current for the Rarita-Schwinger field  $\bar{\psi}_\mu(p', x)$  is defined by writing the scattering amplitude as

$$\begin{aligned}\mathcal{A} &= \frac{1}{2}ig \int d^4x \bar{\psi}_\mu(p', x) \mathcal{J}^\mu(p, x) \quad , \\ \mathcal{J}^\mu(p, x) &= \epsilon^{\mu\eta\nu\rho} \gamma_5 \gamma_\eta A_\nu(x) \psi_\rho(p, x) \quad .\end{aligned}\tag{145}$$

Finally, the source current for the Rarita-Schwinger field  $\psi_\rho(p, x)$  is defined by writing the scattering amplitude as

$$\begin{aligned}\mathcal{A} &= \frac{1}{2}ig \int d^4x \bar{\mathcal{J}}^\rho(p', x) \psi_\rho(p, x) \quad , \\ \bar{\mathcal{J}}^\rho(p', x) &= \bar{\psi}_\mu(p', x) \epsilon^{\mu\eta\nu\rho} \gamma_5 \gamma_\eta A_\nu(x) \quad .\end{aligned}\tag{146}$$

We now show that the three currents that we have just defined are conserved. For the source current  $J^\nu$  for the gauge potential, we have

$$\begin{aligned}\partial_\nu J^\nu &= \bar{\psi}_\mu(p', x) \overleftarrow{D}_\nu \epsilon^{\mu\eta\nu\rho} \gamma_5 \gamma_\eta \psi_\rho(p, x) \\ &\quad + \bar{\psi}_\mu(p', x) \epsilon^{\mu\eta\nu\rho} \gamma_5 \gamma_\eta D_\nu \psi_\rho(p, x) \\ &= 0 \quad ,\end{aligned}\tag{147}$$

where the first and second terms on the right vanish by the Rarita-Schwinger equations for  $\bar{\psi}_\mu(p', x)$  and  $\psi_\rho(p, x)$  respectively. For the source current  $\mathcal{J}^\mu(p, x)$  for the spinor  $\bar{\psi}_\mu(p', x)$ , we have

$$\begin{aligned}D_\mu \mathcal{J}^\mu(p, x) &= \epsilon^{\mu\eta\nu\rho} \gamma_5 \gamma_\eta (\partial_\mu A_\nu(x)) \psi_\rho(p, x) \\ &\quad + \epsilon^{\mu\eta\nu\rho} \gamma_5 \gamma_\eta A_\nu(x) D_\mu \psi_\rho(p, x) \\ &= 0 \quad .\end{aligned}\tag{148}$$

The second term on the right vanishes by the Rarita-Schwinger equation for  $\psi_\rho(p, x)$ , while the first term on the right can be rewritten as

$$\frac{1}{2}\epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta F_{\mu\nu}(x)\psi_\rho(p, x) \quad (149)$$

and vanishes by the secondary constraint of Eq. (6). Finally, for the source current  $\overline{\mathcal{J}}^\rho(p', x)$  for the spinor  $\psi_\rho(p, x)$ , we have

$$\begin{aligned} \overline{\mathcal{J}}^\rho(p', x)\overleftarrow{D}_\rho &= \overline{\psi}_\mu(p', x)\epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta(\partial_\rho A_\nu(x)) \\ &\quad + \overline{\psi}_\mu(p', x)\overleftarrow{D}_\rho\epsilon^{\mu\eta\nu\rho}\gamma_5\gamma_\eta A_\nu(x) \\ &= 0 \quad . \end{aligned} \quad (150)$$

Again, the second term on the right vanishes by the Rarita-Schwinger equation, while the first term on the right vanishes by the secondary constraint of Eq. (6).

Consider now the following three gauge transformations,

$$\begin{aligned} A_\nu(x) &\rightarrow A_\nu(x) + \partial_\nu\Lambda \quad , \\ \psi_\rho(p, x) &\rightarrow \psi_\rho(p, x) + D_\rho\alpha \quad , \\ \overline{\psi}_\mu(p', x) &\rightarrow \overline{\psi}_\mu(p', x) + \overleftarrow{\beta}\overleftarrow{D}_\mu \quad , \end{aligned} \quad (151)$$

with  $\alpha$  and  $\beta$  independent spinorial gauge parameters. From Eqs. (144)-(146), together with Eqs. (147)-(150), we find that these transformations all leave the amplitude  $\mathcal{A}$  invariant,

$$\begin{aligned} \delta_\Lambda\mathcal{A} &= \frac{1}{2}ig \int d^4x (\partial_\nu\Lambda)J^\nu(x) = -\frac{1}{2}ig \int d^4x \Lambda\partial_\nu J^\nu(x) = 0 \quad , \\ \delta_\alpha\mathcal{A} &= \frac{1}{2}ig \int d^4x \overline{\mathcal{J}}^\rho(p', x)D_\rho\alpha = -\frac{1}{2}ig \int d^4x \overline{\mathcal{J}}^\rho(p', x)\overleftarrow{D}_\rho\alpha = 0 \quad , \\ \delta_\beta\mathcal{A} &= \frac{1}{2}ig \int d^4x \overleftarrow{\beta}\overleftarrow{D}_\mu\mathcal{J}^\mu(p, x) = -\frac{1}{2}ig \int d^4x \overleftarrow{\beta}D_\mu\mathcal{J}^\mu(p, x) = 0 \quad . \end{aligned} \quad (152)$$

We next must specify more precisely the structure of the spinor wave functions entering the formula for  $\mathcal{A}$ . Since the gauge field  $A_\nu$  is assumed to vanish in the external region  $|\vec{x}| > R$ , the Rarita-Schwinger wave functions obey free field equations in this region. So for  $|\vec{x}| \gg R$  they can

be taken asymptotically as plane waves at  $t \rightarrow \pm\infty$ ,

$$\begin{aligned}\psi_\mu(p', x) &\sim u_\mu(p') e^{ip' \cdot x} \quad , \quad t \rightarrow +\infty \quad , \\ \psi_\rho(p, x) &\sim u_\rho(p) e^{ip \cdot x} \quad , \quad t \rightarrow -\infty \quad .\end{aligned}\tag{153}$$

With these boundary conditions, the formula for the amplitude takes the final form

$$\mathcal{A} = \frac{1}{2} ig \int d^4x \bar{\psi}_\mu^{(-)}(p', x) \epsilon^{\mu\eta\nu\rho} \gamma_5 \gamma_\eta A_\nu(x) \psi_\rho^{(+)}(p, x) \quad .\tag{154}$$

The out state (-) and in state (+) boundary conditions used here are analogs of the boundary conditions used in the distorted wave Born approximation [21], which the construction of Eq. (154) resembles. Equation (154) then gives an approximation to the matrix element for Rarita-Schwinger scattering by the gauge potential.

Finally, using the gauge invariances of  $\mathcal{A}$ , we can now address the question of counting degrees of freedom. We have imposed one gauge fixing constraint on the Rarita-Schwinger wave functions, given in operator form by Eq. (142), which implies that the wave functions obey analogous conditions of vanishing gauge covariant divergence,

$$\bar{\psi}_\mu(p', x) \overleftarrow{D}^\mu = D^\rho \psi_\rho(p, x) = 0 \quad .\tag{155}$$

To see what further conditions can be imposed, we must examine consistency with the Rarita-Schwinger equation for the operator  $\psi_\mu$ . This is most easily done using the alternative form of the Rarita-Schwinger equation given in Eq. (A6),

$$\gamma^\rho (D_\nu \psi_\rho - D_\rho \psi_\nu) = 0 \quad .\tag{156}$$

Applying  $D^\nu$  from the left to this equation, we get

$$D^\nu D_\nu \gamma^\rho \psi_\rho - \gamma_\rho [D^\nu, D^\rho] \psi_\nu - \gamma^\rho D_\rho D^\nu \psi_\nu = 0 \quad ,\tag{157}$$

which using  $[D^\nu, D^\rho] = gF^{\nu\rho}$  becomes

$$D^\nu D_\nu \gamma^\rho \psi_\rho - g\gamma_\rho F^{\nu\rho} \psi_\nu - \gamma^\rho D_\rho D^\nu \psi_\nu = 0 \quad .\tag{158}$$

When  $D^\nu \psi_\nu = 0$ , this equation simplifies to

$$D^\nu D_\nu \gamma^\rho \psi_\rho = g\gamma_\rho F^{\nu\rho} \psi_\nu \quad .\tag{159}$$

In regions where the gauge field is nonvanishing, the right hand side of this equation is nonzero, and so we cannot impose the additional constraint  $\gamma^\rho \psi_\rho = 0$ . This is not surprising, since where the



fields are nonzero there is *already* a second covariant constraint given by the secondary constraint of Eq. (6), which in terms of the dual field  $\tilde{F}^{\nu\rho} = \frac{1}{2}\epsilon^{\nu\rho\alpha\beta}F_{\alpha\beta}$  takes the form

$$0 = \gamma_\rho \tilde{F}^{\nu\rho} \psi_\nu \quad , \quad (160)$$

which differs in form from the right hand side of Eq. (159) when the fields are not self-dual or anti-self-dual. However, in the external region where the gauge field vanishes, Eq. (159) becomes

$$\partial^\nu \partial_\nu \gamma^\rho \psi_\rho = 0 \quad , \quad (161)$$

and we can then impose  $\gamma^\rho \psi_\rho = 0$  by making a gauge transformation in the external region that preserves the original constraint  $\partial^\rho \psi_\rho = 0$ . This is possible, since the condition for the transformation  $\psi_\rho \rightarrow \psi_\rho + \partial_\rho \alpha$  to preserve  $\partial^\rho \psi_\rho = 0$  is  $\partial^\rho \partial_\rho \alpha = 0$ , which is compatible with the form of Eq. (161). This establishes that in the external region we can impose the two Lorentz covariant conditions  $\partial^\rho \psi_\rho = 0$  and  $\gamma^\rho \psi_\rho = 0$ , which together with the primary constraint of Eq. (5) suffices to give the correct degree of freedom counting for the incoming and outgoing Rarita-Schwinger wave functions [22]. Alternatively, if we are not concerned to maintain manifest Lorentz covariance, we can make a gauge transformation in the external region to the gauge  $\psi_0 = \vec{\nabla} \cdot \vec{\psi} = 0$  used in [23] to enumerate Rarita-Schwinger degrees of freedom.

Note that if one were to construct the Born approximation amplitude, in which the Rarita-Schwinger wave functions in the presence of the gauge field are replaced by plane waves in the interior region where the potential is nonzero, the arguments given above for compatibility of Lorentz covariance with degree of freedom counting would fail. The reason for this is that the spinor source currents would then no longer be conserved, even to zeroth order in the gauge coupling  $g$ , because the free particle plane wave solutions do not obey the secondary constraint of Eq. (6). The non-existence of a satisfactory Born approximation for Rarita-Schwinger photon scattering agrees with restrictions on massless particle three point functions obtained by “on-shell” methods [24]. It is also related to the fact that the extra gauge terms  $\partial_\nu \Omega^\mu$  in Eq. (5.27) of [23] for the free Rarita-Schwinger propagator are an obstacle to setting up a Lippmann-Schwinger equation for Rarita-Schwinger photon scattering, which could be expanded in a perturbation series in  $g$ . To establish compatibility, we have had to use an analog of the distorted wave Born approximation [21], in which the leading approximation to the amplitude is constructed using interacting rather than free fermion wave functions and does not have a gauge-invariant small coupling,  $g \rightarrow 0$  limit.

The results just obtained will play a role in the analysis of closed Rarita-Schwinger fermion loops with gauge field vertices. We conjecture that when fermion propagators corresponding to

the interacting wave functions are used in the loops, the usual vertex conservation properties including the known chiral anomalies will hold, while permitting changes of fermionic gauge of the Rarita-Schwinger fermions in the loops. We plan to pursue this as a further investigation.

## XII. DISCUSSION

To conclude, we see that unlike the massive case, the massless gauged Rarita-Schwinger equation leads to a consistent theory at both the classical and quantized levels, and these statements are already foreshadowed in the equations of [4] when taken to the zero mass limit. Thus, non-Abelian gauging of Rarita-Schwinger fields can be contemplated as part of the anomaly cancelation mechanism in constructing grand unified models.

Our analysis invites a number of extensions:

1. We have derived the path integral formula and the constrained Hamilton equations of motion in  $\Psi_0 = 0$  gauge, and shown positivity of the Dirac bracket in covariant radiation gauge. This is reminiscent of what happens in quantizing non-Abelian gauge theories, where unitarity is manifest in one gauge, and renormalizability in another. In the gauge theory case, one has an apparatus for transforming from one gauge to another, but that remains to be worked out for fermionic gauge transformations of the Rarita-Schwinger fields. The analysis of Sec. 11, where we computed an approximation to Rarita-Schwinger photon scattering and found full fermionic gauge invariance, is evidence that at least on shell, where the equations of motion are obeyed, different gauges are equivalent.
2. The analysis of Sec. 11 shows that to get a nonvanishing approximation to Rarita-Schwinger photon scattering, one has to go beyond the Born approximation to scattering, by using in and out solutions of the scattering equations in the presence of the photon perturbation. (We have assumed the existence of these interacting solutions of the Rarita-Schwinger equations, but this remains to be studied.) This is a consequence of the fact that the secondary constraint  $\omega = 0$ , although involving the gauge fields, is zeroth order in the gauge field coupling  $g$  because it arose from application of  $g^{-1}D_\mu$  to the Rarita-Schwinger equation of motion. Thus a quantum theory of Rarita-Schwinger fields in a gauge field background necessarily has a non-perturbative aspect.
3. A possible exception to the non-perturbative behavior of fermion lines is when the  $\vec{E}$  and  $\vec{B}$

gauge fields are random, since if Eq. (50) is replaced by an average, denoted by AV,

$$\langle \Psi_0 \rangle_{AV} \simeq \left\langle \frac{\vec{Q}}{(\vec{B})^2} \right\rangle_{AV} \cdot \langle \vec{\Psi} \rangle_{AV} \quad , \quad (162)$$

it becomes

$$\langle \Psi_0 \rangle_{AV} \simeq \vec{\sigma} \cdot \langle \vec{\Psi} \rangle_{AV} \quad , \quad (163)$$

which is compatible with  $\langle \Psi_0 \rangle_{AV} = \vec{\sigma} \cdot \langle \vec{\Psi} \rangle_{AV} = 0$ , the customary free Rarita-Schwinger constraints employed in [23]. This heuristic observation suggests that Rarita-Schwinger fields coupled to quantized gauge fields with zero background gauge field may have a perturbative  $g \rightarrow 0$  limit.

4. We have included in our analysis only fermionic constraints. However, the action exponent in Eq. (84) has the form (in  $A_0 = 0$  gauge)

$$\begin{aligned} & \frac{1}{2} \int d^4x \left[ -\vec{\Psi}^\dagger \cdot (\vec{B} + \vec{\sigma} \times \vec{E})(\vec{\sigma} \cdot \vec{B})^{-1} \vec{\sigma} \cdot \vec{D} \times \vec{\Psi} \right. \\ & \quad \left. + \vec{\Psi}^\dagger \cdot \vec{\sigma} \times \vec{D}(\vec{\sigma} \cdot \vec{B})^{-1} (\vec{B} + \vec{\sigma} \times \vec{E}) \cdot \vec{\Psi} + \vec{\Psi}^\dagger \cdot \vec{D} \times \vec{\Psi} - \frac{1}{2} \vec{\Psi}^\dagger \times \vec{\sigma} \cdot \partial_0 \vec{\Psi} \right] \\ & = \int d^4x (\partial_0 \vec{\Psi} \cdot \vec{P} - \text{bosonic constraints}) - \int dt H \quad , \end{aligned} \quad (164)$$

with  $H$  the Hamiltonian obtained from the stress-energy tensor. We have not included the bosonic constraints in our analysis, and we note that they will have fermionic brackets with the fermionic constraints  $\phi_a, \chi_a$ . (For a discussion of bosonic versus fermionic constraints, see [25].)

5. In quantizing, we assumed that the gauge fields  $\vec{A}$  are time independent, so that  $d/dt$  and  $\vec{D}$  commute. As already noted, this assumption can be dropped if the gauge fields are treated as dynamical variables, leading to an extension, yet to be analyzed, of the bracket structure, again involving fermionic brackets.
6. In showing in the Abelian case that there is no superluminal propagation, the inversion of  $\vec{\sigma} \cdot \vec{B}$  to get  $\Psi_0$  only required  $(\vec{B})^2 \neq 0$ . In the non-Abelian case, where  $\vec{B}$  is itself a matrix, the conditions for invertibility are nontrivial and have yet to be analyzed.
7. In demonstrating positivity of the anticommutator, we made essential use of the condition  $\vec{\sigma} \cdot \vec{\Psi} = 0$ . Deriving this from the covariant radiation gauge condition  $\vec{D} \cdot \vec{\Psi} = 0$  assumed the

invertibility of  $\vec{\sigma} \cdot \vec{D}$ , and attainability of covariant radiation gauge assumed the invertibility of  $(\vec{D})^2$ . The conditions for invertibility of these two operators remain to be studied. (The open space index theorems of Callias [26] and Weinberg [26] involve  $\vec{\sigma} \cdot \vec{D} + i\phi$ , with  $\phi$  a scalar field, and so do not give information about the invertibility of  $\vec{\sigma} \cdot \vec{D}$ .)

8. The fact that when gauge fields are present, the gauge conditions  $\Psi_0 = 0$  and  $\vec{D} \cdot \vec{\Psi} = 0$  are not equivalent, has consequence for the study of condensates that give rise to a mass for the initially massless Rarita-Schwinger field. The natural way to form a condensate involving the spin  $\frac{3}{2}$  field  $\psi_\mu$  and a spin  $\frac{1}{2}$  field  $\lambda$  is through a structure of the form  $\bar{\lambda} \gamma^\mu \psi_\mu$ . If  $\Psi_0$  and  $\vec{\sigma} \cdot \vec{\Psi}$  were simultaneously zero, this would translate back in the covariant formalism to  $\gamma^\mu \psi_\mu = 0$ , forbidding such condensate formation. Thus, the possibility of dynamical mass generation for the spin  $\frac{3}{2}$  field depends on the inequivalence that we have demonstrated of the gauge conditions  $\Psi_0 = 0$  and  $\vec{D} \cdot \vec{\Psi} = 0$  in nonzero gauge field backgrounds. An alternative way to see that the gauge condition  $\gamma^\mu \psi_\mu = 0$  cannot be imposed when fields are present is to note that its left chiral reduction is  $\Psi_0 = \vec{\sigma} \cdot \vec{\Psi}$ , which in general conflicts with the secondary constraint  $\vec{\sigma} \cdot \vec{B} \Psi_0 = (\vec{B} + \vec{\sigma} \times \vec{E}) \cdot \vec{\Psi}$ .

### XIII. ACKNOWLEDGEMENTS

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### Appendix A: Notational conventions and useful identities

We follow in general the notational conventions of the book *Supergravity* by Freedman and Van Proeyen [23]. The metric  $\eta_{\mu\nu}$  is  $(-, +, +, +)$  and the Dirac gamma matrices  $\gamma_\mu, \gamma^\mu$  obey the Clifford algebra

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \quad . \quad (\text{A1})$$

They are given in terms of Pauli matrices  $\sigma_j$  by

$$\begin{aligned}\gamma_0 &= -\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \\ \gamma_j &= \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad , \\ \gamma_5 &= i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .\end{aligned}\tag{A2}$$

We also note that

$$\epsilon_{0123} = -\epsilon^{0123} = 1 \quad ,\tag{A3}$$

the left chiral projector  $P_L$  is given by

$$P_L = \frac{1}{2}(1 + \gamma_5) \quad ,\tag{A4}$$

and the spinor  $\bar{\psi}$  is defined in terms of the adjoint spinor  $\psi^\dagger$  by

$$\bar{\psi} = \psi^\dagger i\gamma^0 \quad .\tag{A5}$$

As noted in [23], the Rarita-Schwinger equation of motion can be written in a number of equivalent forms. When ordinary derivatives are replaced by gauge covariant derivatives, these are the vector-spinor equations

$$\begin{aligned}\epsilon^{\mu\eta\nu\rho}\gamma_\eta D_\nu\psi_\rho &= 0 \quad , \\ \gamma^{\eta\nu\rho}D_\nu\psi_\rho &= 0 \quad , \\ \gamma^\rho(D_\nu\psi_\rho - D_\rho\psi_\nu) &= 0 \quad , \\ \gamma^\alpha D_\alpha(D_\sigma\psi_\nu - D_\nu\psi_\sigma) &= \gamma^\rho\left([D_\rho, D_\sigma]\psi_\nu + [D_\nu, D_\rho]\psi_\sigma + [D_\sigma, D_\nu]\psi_\rho\right) \quad ,\end{aligned}\tag{A6}$$

with only the fourth line, which is quadratic in the covariant derivative, involving more than just a substitution  $\partial_\nu \rightarrow D_\nu$  in the formulas of [23]. Using  $\gamma_\eta\gamma^{\eta\nu\rho} = 2\gamma^{\nu\rho}$ , these also imply the spinor equation  $\gamma^{\nu\rho}D_\nu\psi_\rho = 0$ . These formulas play a role in verifying stress-energy tensor conservation, as does the identity [27]

$$0 = \epsilon^{\lambda\sigma\mu\nu}(A_\tau B_\lambda C_\sigma D_\mu E_\nu + A_\nu B_\tau C_\lambda D_\sigma E_\mu + A_\mu B_\nu C_\tau D_\lambda E_\sigma + A_\sigma B_\mu C_\nu D_\tau E_\lambda + A_\lambda B_\sigma C_\mu D_\nu E_\tau) \quad ,\tag{A7}$$

with  $A_\tau, B_\lambda, C_\sigma, D_\mu, E_\nu$  five arbitrary four vectors. This identity follows from

$$0 = \delta_\tau^\alpha \epsilon^{\lambda\sigma\mu\nu} + \delta_\tau^\nu \epsilon^{\alpha\lambda\sigma\mu} + \delta_\tau^\mu \epsilon^{\nu\alpha\lambda\sigma} + \delta_\tau^\sigma \epsilon^{\mu\nu\alpha\lambda} + \delta_\tau^\lambda \epsilon^{\sigma\mu\nu\alpha} \quad , \quad (\text{A8})$$

which is easily verified by noting that  $\lambda, \sigma, \mu, \nu$  must take distinct values from the set  $0, 1, 2, 3$ , and that  $\tau$  must be equal to one of these values.

The fundamental identity for the Pauli matrices is

$$\sigma_a \sigma_b = \delta_{ab} + i\epsilon_{abc} \sigma_c \quad , \quad (\text{A9})$$

with  $\epsilon_{123} = 1$  and with the index  $c$  summed. We repeatedly use the following two identities that can be derived from Eq. (A9), for a general three vector  $\vec{A}$  that is proportional to a unit matrix in the spinor space and so commutes with  $\vec{\sigma}$ ,

$$\begin{aligned} \vec{\sigma} \times (\vec{\sigma} \times \vec{A}) &= -2\vec{A} + i\vec{\sigma} \times \vec{A} \quad , \\ (\vec{A} \times \vec{\sigma}) \times \vec{\sigma} &= -2\vec{A} + i\vec{A} \times \vec{\sigma} \quad . \end{aligned} \quad (\text{A10})$$

Additional useful identities are

$$\begin{aligned} \vec{\sigma} \times \vec{\sigma} &= 2i\vec{\sigma} \quad , \\ \vec{\sigma} \vec{\sigma} \cdot \vec{A} &= \vec{A} - i\vec{\sigma} \times \vec{A} \quad , \\ \vec{\sigma} \cdot \vec{A} \vec{\sigma} &= \vec{A} + i\vec{\sigma} \times \vec{A} \quad , \\ (\vec{\sigma} \times \vec{A}) \cdot \vec{\sigma} &= -2i\vec{\sigma} \cdot \vec{A} \quad , \\ \vec{\sigma} \cdot (\vec{\sigma} \times \vec{A}) &= 2i\vec{\sigma} \cdot \vec{A} \quad , \\ \sigma_a \sigma_b &= 2\left(\delta_{ab} - \frac{1}{2}\sigma_b \sigma_a\right) \quad , \\ \vec{B} = i\vec{A} - \vec{A} \times \vec{\sigma} &\leftrightarrow \vec{A} = \frac{1}{2}(\vec{B} \times \vec{\sigma}) \quad . \end{aligned} \quad (\text{A11})$$

Gauge field covariant derivatives are

$$D_\mu = \partial_\mu + gA_\mu \quad , \quad (\text{A12})$$

with the gauge potential  $A_\mu = A_\mu^A t_A$  and the gauge generators  $t_A$  anti-self-adjoint, and with the components  $A_\mu^A$  self-adjoint. The non-Abelian generators  $t_A$  obey the compact Lie algebra

$$[t_A, t_B] = f_{ABC} t_C \quad ; \quad (\text{A13})$$

in the Abelian case we replace  $t_A$  by  $-i$ . In writing field strengths  $\vec{E}$  and  $\vec{B}$  we pull out an additional factor of  $i$  to make them self-adjoint, so that we have the identities

$$\begin{aligned}\vec{D} \times \vec{D} &= -ig\vec{B} \quad , \\ [\vec{D}, D_0] &= -ig\vec{E} \quad .\end{aligned}\tag{A14}$$

We will also write a right-acting three-vector covariant derivative as  $\vec{D} = \vec{\nabla} + g\vec{A}$ , and define a left-acting three-vector covariant derivative as  $\overleftarrow{D} = \overleftarrow{\nabla} - g\vec{A}$ , so that we have the integration by parts formulas

$$\begin{aligned}\int d^3x A \vec{D}_{\vec{x}} B &= - \int d^3x A \overleftarrow{D}_{\vec{x}} B \quad , \\ \vec{D}_{\vec{x}} \delta^3(\vec{x} - \vec{y}) &= - \delta^3(\vec{x} - \vec{y}) \overleftarrow{D}_{\vec{y}} \quad .\end{aligned}\tag{A15}$$

An analogous definition is used for the operators  $\vec{L}$  and  $\overleftarrow{L}$  which enter the gauge fixing condition.

At the classical level, variables will be either Grassmann even or odd. Irrespective of the Grassmann parity of monomials  $A$  and  $B$ , the adjoint operation is defined by [23]

$$(AB)^\dagger = B^\dagger A^\dagger \quad .\tag{A16}$$

For classical brackets, we follow the convention of Henneaux and Teitelboim [28],

$$[F, G]_C = \left( \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i} \right) + (-)^{\epsilon_F} \left( \frac{\partial^L F}{\partial \theta^\alpha} \frac{\partial^L G}{\partial \pi_\alpha} + \frac{\partial^L F}{\partial \pi_\alpha} \frac{\partial^L G}{\partial \theta^\alpha} \right) \quad ,\tag{A17}$$

with  $\epsilon_F$  the Grassmann parity of  $F$ , with  $\partial^L$  a Grassmann derivative acting from the left, and with  $q^i, p_i$  ( $\theta^\alpha, \pi_\alpha$ ) canonical coordinates and momenta of even (odd) Grassmann parity. Using the classical bracket, the Dirac bracket is constructed from the constraints as in Eq. (96) of the text. To make the transition to quantum theory, the quantum commutator (anticommutator) is defined to be  $i\hbar$  times the corresponding Dirac bracket (with  $\hbar = 1$  in our notation). Classical canonical brackets are always denoted, as above, by a subscript  $C$ , with a subscript  $D$  used for the corresponding Dirac bracket. We use the standard notations  $[A, B] = AB - BA$  for the commutator and  $\{A, B\} = AB + BA$  for the anticommutator.

To calculate the Dirac bracket, we use block inversion of a matrix. Let

$$M = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} ,$$

$$M^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} ,$$
(A18)

with  $A_1, \dots, A_4$  themselves matrices. Then when  $A_4$  is non-singular, the blocks  $B_1, \dots, B_4$  of  $M^{-1}$  are given by

$$\begin{aligned} \Delta &\equiv A_1 - A_2 A_4^{-1} A_3 \quad , \\ B_1 &= \Delta^{-1} \quad , \\ B_2 &= -\Delta^{-1} A_2 A_4^{-1} \quad , \\ B_3 &= -A_4^{-1} A_3 \Delta^{-1} \quad , \\ B_4 &= A_4^{-1} + A_4^{-1} A_3 \Delta^{-1} A_2 A_4^{-1} \quad . \end{aligned}$$
(A19)

Even though the blocks are noncommutative, Eqs. (A18) and (A19) give an inverse that obeys  $M^{-1}M = MM^{-1} = 1$ .

When the constraints  $\phi_a$  and  $\chi_a$  are combined into an 8 element set of constraints  $\kappa_a = \phi_a, \kappa_{a+4} = \chi_a, a = 1, \dots, 4$  then the bracket matrix  $S_{ab}(\vec{x}, \vec{y}) \equiv [\kappa_a(\vec{x}), \kappa_b(\vec{y})]_C$  can be expressed in terms of the matrix  $M_{ab}(\vec{x}, \vec{y})$  of Eq. (74) as

$$S(\vec{x}, \vec{y}) = \begin{pmatrix} 0 & M(\vec{x}, \vec{y}) \\ M^T(\vec{y}, \vec{x}) & 0 \end{pmatrix} ,$$
(A20)

where  $M_{ab}^T(\vec{x}, \vec{y}) = M_{ba}(\vec{x}, \vec{y})$  is the matrix transpose. Defining the inverse  $M^{-1}(\vec{x}, \vec{y})$  that obeys  $\int d^3z M^{-1}(\vec{x}, \vec{z}) M(\vec{z}, \vec{y}) = \int d^3z M(\vec{x}, \vec{z}) M^{-1}(\vec{z}, \vec{y}) = \delta^3(\vec{x} - \vec{y})$ , it is easy to verify that

$$S^{-1}(\vec{x}, \vec{y}) = \begin{pmatrix} 0 & M^{T^{-1}}(\vec{y}, \vec{x}) \\ M^{-1}(\vec{x}, \vec{y}) & 0 \end{pmatrix} .$$
(A21)

## Appendix B: Analysis of the Rarita-Schwinger field in an external Abelian gauge field, without using the covariant radiation gauge constraint: propagation of the longitudinal mode

We continue here the analysis begun in Sec. 5, but now without using the covariant radiation gauge constraint  $\vec{D} \cdot \vec{\Psi} = 0$ , to analyze propagation of the longitudinal mode that is eliminated by



gauge fixing. We must now solve for  $C_3^{\uparrow,\downarrow}$  starting from Eq. (64) with  $C_{1,2} = 0$ , so the third line of Eq. (64) simplifies to

$$\begin{aligned} 0 &= (\vec{B})^2 \Omega C_3^{\uparrow,\downarrow} - K(Q_3 C_3)^{\uparrow,\downarrow} \quad , \\ Q_3 &= B_1 E_2 - B_2 E_1 + B_3 \vec{\sigma} \cdot (\vec{B} + i\vec{E}) - i\vec{B} \cdot \vec{E} \sigma_3 \quad . \end{aligned} \tag{B1}$$

Writing this as

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} C_3^{\uparrow} \\ C_3^{\downarrow} \end{pmatrix} \quad , \tag{B2}$$

we find for the matrix elements

$$\begin{aligned} U_{11} &= (\vec{B})^2 \Omega - K[B_1 E_2 - B_2 E_1 - i(B_1 E_1 + B_2 E_2) + B_3^2] \quad , \\ U_{22} &= (\vec{B})^2 \Omega - K[B_1 E_2 - B_2 E_1 + i(B_1 E_1 + B_2 E_2) - B_3^2] \quad , \\ U_{12} &= -KB_3[B_1 + iE_1 - i(B_2 + iE_2)] \quad , \\ U_{21} &= -KB_3[B_1 + iE_1 + i(B_2 + iE_2)] \quad . \end{aligned} \tag{B3}$$

The equation  $0 = \det(U) = U_{11}U_{22} - U_{12}U_{21}$  reduces, after dividing by an overall factor of  $(\vec{B})^2$ , to

$$0 = (\vec{B})^2 \Omega^2 - 2\Omega K(B_1 E_2 - B_2 E_1) + K^2(E_1^2 + E_2^2 - B_3^2) \quad , \tag{B4}$$

with the solution

$$\begin{aligned} \frac{\Omega}{K} &= \frac{X \pm Y^{1/2}}{(\vec{B})^2} \quad , \\ X &= B_1 E_2 - B_2 E_1 \quad , \\ Y &= (B_1 E_2 - B_2 E_1)^2 - (\vec{B})^2(E_1^2 + E_2^2 - B_3^2) \quad . \end{aligned} \tag{B5}$$

The analysis of the solutions of Eqs. (B4) and (B5) divides into two cases, according to whether the roots of Eq. (B5) are both real, or both complex. The roots are both complex if

$$(B_1 E_2 - B_2 E_1)^2 < (\vec{B})^2(E_1^2 + E_2^2 - B_3^2) \quad , \tag{B6}$$

which can be rearranged algebraically to the form

$$[(\vec{B})^2 - (E_1^2 + E_2^2)]B_3^2 < (B_1^2 + B_2^2)(E_1^2 + E_2^2) \cos^2 \phi \quad , \quad (\text{B7})$$

where we have written

$$\begin{aligned} B_1 E_2 - B_2 E_1 &= (B_1^2 + B_2^2)^{1/2} (E_1^2 + E_2^2)^{1/2} \sin \phi \quad , \\ B_1 E_1 + B_2 E_2 &= (B_1^2 + B_2^2)^{1/2} (E_1^2 + E_2^2)^{1/2} \cos \phi \quad . \end{aligned} \quad (\text{B8})$$

Since the right hand side of Eq. (B7) is non-negative, when the left hand side is negative the inequality is satisfied, and both roots are complex. Hence a necessary (but not sufficient) condition for both roots to be real is

$$(\vec{B})^2 - (E_1^2 + E_2^2) > 0 \quad . \quad (\text{B9})$$

### 1. The hyperbolic case: both roots real

When both roots are real, Eq. (B1) describes the hyperbolic case of propagating waves. Introducing the velocity  $V = \Omega/K$ , Eq. (B4) can be written as

$$0 = (\vec{B})^2 V^2 - 2V(B_1 E_2 - B_2 E_1) + E_1^2 + E_2^2 - B_3^2 \quad , \quad (\text{B10})$$

which can be rearranged algebraically to the form

$$[(B_1^2 + B_2^2)^{1/2} - (E_1^2 + E_2^2)^{1/2}]^2 + (\vec{B})^2 (V^2 - 1) = 2(B_1^2 + B_2^2)^{1/2} (E_1^2 + E_2^2)^{1/2} (V \sin \phi - 1) \quad . \quad (\text{B11})$$

Let us now assume that  $V^2 > 1$ , and show that this leads to a contradiction. When  $V^2 > 1$ , the left hand side of Eq. (B11) is nonnegative, which implies that  $V \sin \phi$  on the right must be nonnegative, and so can be replaced by its absolute value. Hence the right hand side of Eq. (B11) obeys the inequality

$$2(B_1^2 + B_2^2)^{1/2} (E_1^2 + E_2^2)^{1/2} (V \sin \phi - 1) = 2(B_1^2 + B_2^2)^{1/2} (E_1^2 + E_2^2)^{1/2} (|V \sin \phi| - 1) \leq 2(\vec{B})^2 (|V| - 1) \quad , \quad (\text{B12})$$

where we have used Eq. (B9). But the left hand side of Eq. (B11) obeys the inequality

$$[(B_1^2 + B_2^2)^{1/2} - (E_1^2 + E_2^2)^{1/2}]^2 + (\vec{B})^2 (V^2 - 1) \geq (\vec{B})^2 (|V| + 1)(|V| - 1) > 2(\vec{B})^2 (|V| - 1) \quad , \quad (\text{B13})$$

which is a contradiction, since a real number cannot be strictly less than itself. Hence we must have  $V^2 \leq 1$ , and there is no superluminal propagation.

## 2. The elliptic case: both roots complex

When both roots are complex, Eq. (B1) describes the elliptic case in which there are no propagating waves; when a propagating wave enters an elliptic region from a hyperbolic one it will be damped to zero amplitude. However, in the case of weak damping one can still define a wave velocity and ask what its magnitude is. When both roots are imaginary, Eq. (B5) takes the form

$$\begin{aligned}\frac{\Omega}{K} &= \frac{X \pm i(-Y)^{1/2}}{(\vec{B})^2} \quad , \\ X &= B_1 E_2 - B_2 E_1 \quad , \\ -Y &= -(B_1 E_2 - B_2 E_1)^2 + (\vec{B})^2 (E_1^2 + E_2^2 - B_3^2) \quad .\end{aligned}\tag{B14}$$

Regarding  $\Omega$  as real and the wave number  $K$  as complex, the effective propagation velocity has the magnitude

$$|V_{\text{eff}}| = \left| \frac{\Omega}{K_R} \right| = \frac{X^2 - Y}{(\vec{B})^2 |X|} = \frac{E_1^2 + E_2^2 - B_3^2}{|B_1 E_2 - B_2 E_1|} \quad .\tag{B15}$$

The condition for weak damping is  $-Y \ll X^2$ , which can be rewritten as

$$(\vec{B})^2 (E_1^2 + E_2^2 - B_3^2) \ll 2(B_1 E_2 - B_2 E_1)^2 \quad ,\tag{B16}$$

and implies

$$|V_{\text{eff}}| \ll \frac{2|B_1 E_2 - B_2 E_1|}{(\vec{B})^2} \leq \frac{2|\vec{E}|}{|\vec{B}|} \quad .\tag{B17}$$

Hence as long as  $2|\vec{E}|$  is not much larger than  $|\vec{B}|$ , which is required by the vacuum stability condition  $|\vec{E}| < |\vec{B}|$ , the damped wave propagation velocity is subluminal.

## Appendix C: Construction of the projector $R_{ij}(\vec{x}, \vec{y})$

Since there are only two  $\phi_a$  constraints and two  $\chi_a$  constraints, we index them  $a = 1, 2$  rather than  $a = 3, 4$  as in the text, and use the invariance of  $R_{ij}(\vec{x}, \vec{y})$  under changing the linear combination of the  $\chi_a$  constraints. We start from the constraint set

$$\begin{aligned}\phi_1 &= \vec{\sigma} \times \vec{D} \cdot \vec{\Psi} \quad , \quad \chi_1 = \vec{P} \cdot \overleftarrow{D} \quad , \\ \phi_2 &= \vec{D} \cdot \vec{\Psi} \quad , \quad \chi_2 = \vec{P} \cdot \vec{\sigma} \times \overleftarrow{D} \quad .\end{aligned}\tag{C1}$$

For the bracket matrix

$$M_{ab}(\vec{x}, \vec{y}) = [\phi_a(\vec{x}), \chi_b(\vec{y})]_C = \begin{pmatrix} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{pmatrix} , \quad (\text{C2})$$

we find the matrix elements

$$\begin{aligned} \hat{\mathcal{A}} &= -ig\vec{\sigma} \cdot \vec{B}\delta^3(\vec{x} - \vec{y}) \quad , \\ \hat{\mathcal{B}} &= (2(\vec{D}_{\vec{x}})^2 + g\vec{\sigma} \cdot \vec{B})\delta^3(\vec{x} - \vec{y}) = \delta^3(\vec{x} - \vec{y})(2(\overleftarrow{D}_{\vec{y}})^2 + g\vec{\sigma} \cdot \vec{B}) \quad , \\ \hat{\mathcal{C}} &= (\vec{D}_{\vec{x}})^2\delta^3(\vec{x} - \vec{y}) = \delta^3(\vec{x} - \vec{y})(\overleftarrow{D}_{\vec{y}})^2 \quad , \\ \hat{\mathcal{D}} &= ig\vec{\sigma} \cdot \vec{B}\delta^3(\vec{x} - \vec{y}) \quad . \end{aligned} \quad (\text{C3})$$

We write the inverse matrix  $M^{-1}(\vec{z}, \vec{w})$  as

$$\begin{pmatrix} \hat{\mathcal{F}} & \hat{\mathcal{G}} \\ \hat{\mathcal{H}} & \hat{\mathcal{I}} \end{pmatrix} , \quad (\text{C4})$$

which obeys

$$\begin{pmatrix} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{pmatrix} \begin{pmatrix} \hat{\mathcal{F}} & \hat{\mathcal{G}} \\ \hat{\mathcal{H}} & \hat{\mathcal{I}} \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{F}} & \hat{\mathcal{G}} \\ \hat{\mathcal{H}} & \hat{\mathcal{I}} \end{pmatrix} \begin{pmatrix} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \quad (\text{C5})$$

In terms of the inverse matrix, the projector  $R_{ij}(\vec{x}, \vec{w})$  is given by (with internal symmetry indices suppressed)

$$\begin{aligned} R_{ij}(\vec{x}, \vec{w}) &= \delta_{ij}\delta^3(\vec{x} - \vec{w})1 \\ &+ D_{\vec{x}i}\hat{\mathcal{F}}(\vec{x} - \vec{w})(\vec{\sigma} \times \overleftarrow{D}_{\vec{w}})_j + D_{\vec{x}i}\hat{\mathcal{G}}(\vec{x} - \vec{w})\overleftarrow{D}_{\vec{w}j} \\ &+ (\vec{\sigma} \times D_{\vec{x}})_i\hat{\mathcal{H}}(\vec{x} - \vec{w})(\vec{\sigma} \times \overleftarrow{D}_{\vec{w}})_j + (\vec{\sigma} \times D_{\vec{x}})_i\hat{\mathcal{I}}(\vec{x} - \vec{w})\overleftarrow{D}_{\vec{w}j} \quad . \end{aligned} \quad (\text{C6})$$

From this expression, we find

$$D_{\vec{x}i}R_{ij}(\vec{x}, \vec{w}) = R_{ij}(\vec{x}, \vec{w})\overleftarrow{D}_{\vec{w}j} = (\vec{\sigma} \times D_{\vec{x}})_i R_{ij}(\vec{x}, \vec{w}) = R_{ij}(\vec{x}, \vec{w})(\vec{\sigma} \times \overleftarrow{D}_{\vec{w}})_j = 0 \quad . \quad (\text{C7})$$

In verifying these, it is not necessary to evaluate the inverse matrix; instead, after contracting on the vector index  $i$  or  $j$  one expresses the resulting pre- or post- factor in terms of  $\hat{\mathcal{A}}, \dots, \hat{\mathcal{D}}$  and then uses the algebraic relations following from multiplying out the matrices in Eq. (C5). Finally,

contracting

$$\begin{aligned}\vec{\sigma} \cdot \vec{D}_{\vec{x}} \sigma_i &= (D_{\vec{x}} + i\vec{\sigma} \times \vec{D}_{\vec{x}})_i \quad , \\ \sigma_j \vec{\sigma} \cdot \overleftarrow{D}_{\vec{w}} &= (\overleftarrow{D}_{\vec{w}} - i\vec{\sigma} \times \overleftarrow{D}_{\vec{w}})_j \quad ,\end{aligned}\tag{C8}$$

with  $R_{ij}(\vec{x}, \vec{w})$ , we conclude that

$$\sigma_i R_{ij}(\vec{x}, \vec{y}) = R_{ij}(\vec{x}, \vec{y}) \sigma_j = 0 \quad ,\tag{C9}$$

when  $\vec{\sigma} \cdot \vec{D}$  is invertible.

As a consequence of Eqs. (127) and (C7),  $\tilde{\Psi}_i(\vec{x})$  is invariant under the transformations

$$\begin{aligned}\vec{\Psi} &\rightarrow \vec{\Psi} + \vec{D}\epsilon \quad , \\ \vec{\Psi} &\rightarrow \vec{\Psi} + \vec{\sigma} \times \vec{D}\epsilon \quad .\end{aligned}\tag{C10}$$

The first of these implies that the canonical anticommutation relations are invariant under infinitesimal Rarita-Schwinger gauge transformations starting from covariant radiation gauge.

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